CS 336

1. The important issue is the logic you used to arrive at your answer.
2. Use extra paper to determine your solutions then neatly transcribe them onto these sheets.
3. Do not submit the scratch sheets. However, all of the logic necessary to obtain the solution should be on these sheets.
4. Comment on all logical flaws and omissions and enclose the

comments in boxes

1. [10] How many five digit decimal numbers (without leading zeros) include at least one digit of 3 or 5?

We solve this by computing a difference. There are 10^5 five digit decimal numbers. Of these 10^4 have a leading zero and 8^5 have no 3 and no 5. There are 8^4 five digit decimal numbers with both a leading zero and no 3 and no 5. Thus, there are $10^4+8^5-8^4$ five digit decimal numbers with a leading zero or having no 3 and no 5, and finally $10^5-(10^4+8^5-8^4)=10^5-10^4-8^5+8^4$ five digit decimal numbers (without leading zeros) including at least one digit of 3 or 5. (An alternative approach goes directly to the algebraically equivalent solution $9\cdot 10^4-7\cdot 8^4$.)

2. [10] Suppose 13 card hands are to be drawn from a regular 52-card deck except that the numbers on the cards are ignored and only the suits matter. Thus, there are 13 spades, 13 hearts etc. . Also, the order of the cards is irrelevant. How many different hands exist?

Thirteen cards are to drawn from four suits allowing repetition and ignoring order. The number of such hands then is $\binom{13+4-1}{4-1}=\binom{16}{3}$.

3. a. [10] Using a combinatorial argument, prove that for $n\ge 1$ and $k\ge 1$:

$$n^k-n^{k-1}=(n-1)n^{k-1}$$

Consider arrays of length k selected from a set $\{a_1,a_2,\ldots,a_n\}$ in which the first element cannot be a_1. For the left side, there are n^k total arrays without the restriction and n^{k-1} arrays that have a_1.as the first element. Thus, there are n^k-n^{k-1} arrays in which the first element is not a_1. Alternatively, there are $n-1$ non-a_1 options for the first elements and n options for the remaining $k-1$ elements, giving $(n-1)n^{k-1}$. We may conclude that $n^k-n^{k-1}=(n-1)n^{k-1}$.

b. [10] Using a combinatorial argument, prove that for $m \geq n \geq p \geq 0$:

$$\binom{m}{n}\binom{n}{p} = \binom{m}{p}\binom{m-p}{n-p}$$

Consider selecting two distinct subsets, A and B, of cardinalities $n-p$ and p, respectively, from a set C of cardinality m. For the left side, there are $\binom{m}{n}$ ways to select $A \cup B$ from C, then $\binom{n}{p}$ ways to select B from $A \cup B$. The remaining elements of $A \cup B$ become A. Thus, there are $\binom{m}{n}\binom{n}{p}$ such decompositions. Alternatively, we may select the elements of B first in $\binom{m}{p}$ ways and then select the $n-p$ elements of A from the remaining $m-p$ elements of $C \square B$. This can be done in $\binom{m-p}{n-p}$ ways, so there are there are $\binom{m}{p}\binom{m-p}{n-p}$ such selections and this must equal $\binom{m}{n}\binom{n}{p}$.

4. a. [5] Consider all distinctly appearing arrangements of the letters of TALLAHASSEE equally likely. What is the probability that such an arrangement spells TALLAHASSEE or EESSAHALLAT?

There are 3 as, 2es, 2 ls, 2 ss, 1 h and 1 t, so there are $\binom{11}{3\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1}$ equally likely arrangements and thus the probability of spelling TALLAHASSEE or EESSAHALLAT is $2/\binom{11}{3\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1}$.

b. [5] What is the probability that such an arrangement spells TALLAHASSEE or EESSAHALLAT given that the two ls appear together?

Of the $\binom{11}{3\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1}$ equally likely arrangements, $\binom{10}{3\ \ 2\ \ 2\ \ 1\ \ 1\ \ 1}$ have the two ls together. Thus, the probability of spelling TALLAHASSEE or EESSAHALLAT is $2/\binom{10}{3\ \ 2\ \ 2\ \ 1\ \ 1\ \ 1}$..

5. [15] Prove: If A and B are countably infinite so is $A \times B$.

Since A and B are countably infinite, there exist mappings $f : \mathbb{N} \xrightarrow[onto]{1-1} A$ and $g : \mathbb{N} \xrightarrow[onto]{1-1} B$. For $i \in \mathbb{N}$, let $C_i = \{f(i)\} \times B$ and define $h_i : \mathbb{N} \to C_i$ by $h_i(j) = (f(i), g(j))$. Since g is onto, for any $b \in B$, there exitst $j \in \mathbb{N}$ so that $g(j) = b$. Thus for any $(f(i), b) \in C_i$ there exists $j \in \mathbb{N}$ so that $h_i(j) = (f(i), b)$. So h_i is onto. It is also one-to-one since for $j_1 \neq j_2, g(j_1) \neq g(j_2)$, since g is one-to one, thus for $j_1 \neq j_2, h_i(j_1) = (f(i), g(j_1)) \neq (f(i), g(j_2)) = h_i(j_2)$. We conclude that for $i \in \mathbb{N}$, C_i is countably infinite. By Theorem 10, $\bigcup_{i \in \mathbb{N}} C_i$ is countably infinite, but $\bigcup_{i \in \mathbb{N}} C_i = A \times B$.

6. [10] You are given that if sets A and B are finite, then so is $A \cup B$. Using induction, prove for $k \geq 1$, if $A_1, A_2, ,..., A_k$ are finite, then so is $\bigcup_{i=1}^{k} A_i$.

For $k = 1$, $\bigcup_{i=1}^{k} A_i = A_1$, and A_1 is finite. Now assume that if $A_1, A_2, ,..., A_k$ are finite, then so is $\bigcup_{i=1}^{k} A_i$, and consider $\bigcup_{i=1}^{k+1} A_i$, assuming $A_1, A_2, ,..., A_k, A_{k+1}$ are finite. We have $\bigcup_{i=1}^{k+1} A_i = \bigcup_{i=1}^{k} A_i \cup A_{k+1}$: the first is finite from the inductive hypothesis, the second is finite from the assumption, thus the union is also finite. By induction, we have for $k \geq 1$, if $A_1, A_2, ,..., A_k$ are finite, then so is $\bigcup_{i=1}^{k} A_i$.

7. [15] Over the set of real-valued functions defined on the natural numbers, define the relation Θ by $f \Theta g$ if and only if $f = O(g)$ and $g = O(f)$. Prove Θ is an equivalence relation.

For reflexivity, we must show $f \Theta f$. However, for $N = 0, M = 1$, we have $n \geq N$ implies $|f(n)| \leq M |f(n)|$, so $f = O(f)$ and $f \Theta f$. For symmetry, assume $f \Theta g$; but then $f = O(g)$ and $g = O(f)$ so $g \Theta f$. Finally for transitivity, assume $f \Theta g$ and $g \Theta h$. We then have in particular $f = O(g)$ and $g = O(h)$. Thus there exist $N_1, M_1, N_2,$ and M_2 so that for $n \geq N_1, |f(n)| \leq M_1 |g(n)|,$ and for $n \geq N_2, |g(n)| \leq M_2 |h(n)|$. Thus for $N_3 = \max\{N_1, N_2\}$ and $M_3 = M_1 M_2$, we have for $n \geq N_3, |f(n)| \leq M_1 |g(n)| \leq M_1 M_2 |h(n)| = M_1 M_2 |h(n)|,$ so $f = O(h)$. However, since $f \Theta g$ and $g \Theta h$, we also have $h = O(g)$ and $g = O(f)$, so we may also conclude $h = O(g)$. Since $f = O(h)$ and $h = O(g)$, we have $f \Theta h$. Having proved that Θ is reflexive, symmetric, and transitive, we conclude that it is an equivalence relation.

8. [10] . Prove that if $1 < a$, then $n + \frac{1}{n} = o(n^a)$. (Hint: First bound $n + \frac{1}{n}$ by something that is also $o(n^a)$.)

Given $\varepsilon > 0$, let $N = (\frac{2}{\varepsilon})^{1/(a-1)}$. For $n \geq N$ we have $n^{a-1} \geq \frac{2}{\varepsilon}$, thus $n^{1-a} \leq \frac{\varepsilon}{2}$ and $2n \leq \varepsilon n^a$. Since $|n + \frac{1}{n}| \leq 2n \leq \varepsilon |n^a|$, we conclude $n + \frac{1}{n} = o(n^a)$.

9. [10]. Prove the code below is correct with respect to precondition "$(x = a) \wedge (y = b)$" and postcondition "$(b > 2a^8) \vee (1 \leq y)$". Assume x, and y are integer variables. (Hint: To simplify the process, keep the actual postcondition in mind – it may be much weaker than the strongest postcondition.)

```
x := x*x
x := x*x
x := x*x
if y > 3*x then
        y := 0
        if x > 100 then
                x : = 15*x-1
        else
                x := -x
        endif
else
        y := 1
        x := 2*x
endif
```

	$(x = a) \wedge (y = b)$
x := x*x	$(x = a^2) \wedge (y = b)$
x := x*x	$(x = a^4) \wedge (y = b)$
x := x*x	$(x = a^8) \wedge (y = b)$
if y > 3*x **then**	$(x = a^8) \wedge (y = b) \wedge (y > 3x)$
	$(x = a^8) \wedge (y = b) \wedge (b > 3a^8)$
y := 0	$(x = a^8) \wedge (y = 0) \wedge (b > 3a^8)$
if x > 100 **then**	$(x = a^8) \wedge (y = 0) \wedge (b > 3a^8) \wedge (x > 100)$
	$(x = a^8) \wedge (y = 0) \wedge (b > 3a^8) \wedge (a^8 > 100)$
x : = 15*x-1	$(x = 15a^8 - 1) \wedge (y = 0) \wedge (b > 3a^8) \wedge (a^8 > 100)$
	$b > 3a^8$
else	$(x = a^8) \wedge (y = 0) \wedge (b > 3a^8) \wedge (x \leq 100)$

______________ $(x=a^8)\wedge(y=0)\wedge(b>3a^8)\wedge(a^8\le 100)$

x := -x__________ $(x=-a^8)\wedge(y=0)\wedge(b>3a^8)\wedge(a^8\le 100)$

__________ $b>3a^8$

endif______________ $b>3a^8$

else______________________ $(x=a^8)\wedge(y=b)\wedge(y<3x)$

y := 1______________ $(x=a^8)\wedge(y=1)\wedge(y'<3x)$

x := 2*x______________ $(x=2a^8)\wedge(y=1)\wedge(y'<3x)$

______________ $y=1$

endif______________________ $(b>3a^8)\vee(1=y)$

__________________________ $(b>2a^8)\vee(1\le y)$

10. [10] The code below purports to compute the quotient and remainder of two given positive integers. Prove the code is partially correct with respect to precondition "$(x>0)\wedge(y>0)$" and postcondition "$(y=d\cdot x+r)\vee(0\le r<y)$" (assume x, and y are integer variables.):

```
d := 0
r := x
while r ≥ y do
        r := r-y
        d : d+1
endwhile
```

Be explicit about your loop invariant: I = $(y=d\cdot x+r)\wedge(0\le r)$

__________________ $(x>0)\wedge(y>0)$

d := 0______________ $(x>0)\wedge(y>0)\wedge(d=0)$

r := x ______________ $(x>0)\wedge(y>0)\wedge(d=0)\wedge(r=x)$

______________ $(y=d\cdot x+r)\wedge(0\le r)$

while r ≥ y **do**______ $(y=d\cdot x+r)\wedge(0\le r)\wedge(r\ge y)$

r := r-y________ $(y=d\cdot x+r')\wedge(0\le r')\wedge(r'\ge y)\wedge(r=r'-y)$

______ $(y=d\cdot x+r+y)\wedge(0\le r)$

d : d+1______ $(y=d'\cdot x+r+y)\wedge(0\le r)\wedge(d=d'+1)$

______ $(y=d\cdot x+r)\wedge(0\le r)$

endwhile __ $(y=d\cdot x+r)\wedge(0\le r)\wedge(r<y)$

__________________ $(y=d\cdot x+r)\vee(0\le r<y)$

b. [5] Prove that the loop terminates.

If we consider the expression $y-r$:

while r ≥ y **do**
r := r-y________ $y-r<y'-r'$
d : d+1
endwhile

we see that it is a monotonically decreasing sequence of integers, therefore eventually it is negative and the loop terminates.

11. [10] Assuming x, y, and z are integer variables and are defined, determine the weakest precondition with respect to the postcondition

"$(x>0)\wedge(z>y)$":

if ((x > y) ∨ (y > z)) **then**
x: = x+z
y: = y-x
else
z := y
x := z-x
endif

State your answer using only relational operators, $\wedge$, and $\vee$.

Notice first that

wp (x: = x+z; y: = y-x, $(x>0)\wedge(z>y)$)
= **wp** (x: = x+z;, **wp**(y: = y-x, $(x>0)\wedge(z>y)$))
= **wp** (x: = x+z;, $(x>0)\wedge(z>y-x)$)
= $(x+z>0)\wedge(z>y-(x+z))$
= $(x+z>0)\wedge(x+2z>y)$

and

wp (z := y; x := z-x, $(x>0)\wedge(z>y)$)
= wp (z := y; **wp**(x := z-x, $(x>0)\wedge(z>y)$))
= wp (z := y; $(z-x>0)\wedge(z>y)$)
=$(z-x>0)\wedge(y>y)$
=$(z-x>0)\wedge$ **false**
= **false**

So

wp(if ((x > y) ∨ (y > z)) **then** x: = x+z; y: = y-x **else** z := y; x := z-x **endif,** $(x>0)\wedge(z>y)$)
= ($\neg$((x > y) ∨ (y > z))$\vee$ **wp** (x: = x+z; y: = y-x, $(x>0)\wedge(z>y)$))

$\wedge\ (((x > y)\ \vee\ (y > z))\ \vee\ \mathbf{wp}\ (z := y;\ x := z\text{-}x,\ (x > 0) \wedge (z > y)))$

$= (\neg((x > y)\ \vee\ (y > z)) \vee\ ((x + z > 0) \wedge (x + 2z > y))$

$\wedge\ (((x > y)\ \vee\ (y > z))\ \vee\ \mathbf{false})$

$= (((x \le y)\ \wedge\ (y \le z)) \vee\ ((x + z > 0) \wedge (x + 2z > y)) \wedge ((x > y)\ \vee\ (y > z))$

$= ((x + z > 0) \wedge (x + 2z > y)) \wedge ((x > y)\ \vee\ (y > z))$

12. a.[2] Translate the assertion "$m = \max\{x, y\}$" into an equivalent form using only relational and logical operators.

$((m = x) \vee (m = y)) \wedge ((m \ge x) \wedge (m \ge y))$

b. [10] Determine the weakest precondition with respect to the postcondition "$m = \max\{x, y\}$" for the following (assume y and x are integer variables and are defined). Substitute your answer to part a for "$m = \max\{x, y\}$".

if $((x > y) \wedge (x > 0))$ **then**
    m := x
else
    m := y
endif

First

$\mathbf{wp}(m := x,\ ((m = x) \vee (m = y)) \wedge ((m \ge x) \wedge (m \ge y)))$

$= ((x = x) \vee (x = y)) \wedge ((x \ge x) \wedge (x \ge y))$

$= (x \ge y)$

and

$\mathbf{wp}(m := y,\ ((m = x) \vee (m = y)) \wedge ((m \ge x) \wedge (m \ge y)))$

$= ((y = x) \vee (y = y)) \wedge ((y \ge x) \wedge (y \ge y))$

$= (y \ge x)$

so

$\mathbf{wp}(\mathbf{if}\ ((x > y) \wedge (x > 0))\ \mathbf{then}\ m := x\ \mathbf{else}\ m := y\ \mathbf{endif},$

$((m = x) \vee (m = y)) \wedge ((m \ge x) \wedge (m \ge y)))$

$= (\neg((x > y) \wedge (x > 0))\ \vee\ (x \ge y)) \wedge (((x > y) \wedge (x > 0))\ \vee\ (y \ge x))$

$= (((x \le y) \vee (x > 0))\ \vee\ (x \ge y)) \wedge (((x > y) \wedge (x > 0))\ \vee\ (y \ge x))$

$= \mathbf{true} \wedge (((x > y) \wedge (x > 0))\ \vee\ (y \ge x))$

$= ((x > y) \wedge (x > 0))\ \vee\ (y \ge x)$

$= (x > 0)\ \vee\ (y \ge x)$

. c.[3] Is the code correct with respect to the precondition "$x^2 < 5$" (i.e. does "$x^2 < 5$" imply your weakest precondition)?

No - "$x^2 < 5$" does not imply "$(x > 0) \vee (y \ge x)$" since for x = 0 and y = -1 we have $x^2 < 5$ but not $(x > 0) \vee (y \ge x)$.