Some Examples of Proof by Induction

1. By induction, prove that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for $n \geq 0$.

Proof:

For $n \geq 0$, let $P(n)$ = "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$".

Basis step: $P(0)$ is true since $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$.

Inductive step: For $n \geq 0$, $P(n) \Rightarrow P(n+1)$, since if $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, then

$$\begin{aligned}\sum_{i=0}^{n+1} i &= \sum_{i=0}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}.\end{aligned}$$

2. By induction, for $n \geq 1$, prove that if the plane cut by n distinct lines, the interior of the regions bounded by the lines can be colored with red and black so that no two regions sharing a common line segment as a boundary will be colored identically.

Proof:

For $n \geq 1$, let $P(n)$ = "if the plane cut by n distinct lines, the interior of the regions bounded by the lines can be colored with red and black so that no two regions sharing a common line segment as a boundary will be colored identically".

Basis step: $P(1)$ is true since if the plane cut by one line, two regions are formed. One may be colored red and the other black. Thus the two regions are colored differently.

Inductive step: For $n \geq 1$, we shall prove $P(n) \Rightarrow P(n+1)$. Given the plane cut by $n+1$ distinct lines, select one line and remove it. The plane is then cut by n distinct lines. By the inductive hypothesis, the interior of the regions bounded by the lines can be colored with red and black so that no two regions sharing a common line segment as a boundary will be colored identically. Now reintroduce the $n+1^{st}$ line and reverse the color of all regions on one side of the line. All regions have line segments that either lie on the $n+1^{st}$ line or one of the other n lines and not both since the lines are distinct. If the boundary line segment lies on the $n+1^{st}$ line, then the $n+1^{st}$ line has cut across a region. Since the region previously had a single color and now has had one of those colored altered, the two regions have different color across this line segment. If the boundary line segment does not lie on the $n+1^{st}$ line, then it lies on one of the other n lines. Furthermore it lies one one side or the other of the $n+1^{st}$ line. Since the regions on either side of the segement previously had different colors, they still do because either they were unchanged after the introduction of the $n+1^{st}$ line or they were both reversed $n+1^{st}$ line. In all cases, the interior of the regions bounded by the $n+1$ lines can be colored with red and black so that no two regions sharing a common line segment as a boundary will be colored identically.

3. By induction, prove that $n^2 \le 2^n$ for $n \ge 4$.

Proof:

For $n \ge 4$, let $P(n)$ = "$n^2 \le 2^n$".
Basis step: $P(4)$ is true since $4^2 = 16 \le 2^4$..
Inductive step: For $n \ge 4$, $P(n) \Rightarrow P(n+1)$, since if $n^2 \le 2^n$, then

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\le n^2 + 2n + n \\ &\le n^2 + 3n \\ &\le n^2 + n \cdot n \\ &\le 2n^2 \\ &\le 2 \cdot 2^n = 2^{n+1}.\end{aligned}$$

4. By induction, prove that the product of any n odd integers is odd for $n \ge 1$.

Proof:

For $n \ge 4$, let $P(n)$ = "the product of any n odd integers is odd".
Basis step: $P(1)$ is true since the product of any single odd integers is itself - which is odd.
Inductive step: For $n \ge 1$, we shall prove $P(n) \Rightarrow P(n+1)$. Prior to that we shall prove a small

Lemma:

The product of two odd integers is odd.

Proof of Lemma:

Let the two odd integers be $2k+1$ and $2j+1$. Their product is
$(2k+1)(2j+1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$ which is odd.

Let the product be represented by $\prod_{i=1}^{n+1} m_i$. The inductive hypothesis guarantees that $\prod_{i=1}^{n} m_i$ is odd. Thus we have $\prod_{i=1}^{n+1} m_i = \prod_{i=1}^{n} m_i \cdot m_{n+1}$ which is the product of two odd integers which, according to our lemma, is odd.

5. By induction, for $n \geq 3$, prove the sum of the interior angles of a convex polygon of n vertices is $(n-2)p$.

Proof:

For $n \geq 3$, let $P(n)$ = "the sum of the interior angles of a convex polygon of n vertices is $(n-2)p$ ".
Basis step: $P(3)$ is true since the sum of the interior angles of a triangle is $p = (3-2)p$.
Inductive step: For $n \geq 3$, we shall prove $P(n) \Rightarrow P(n+1)$. Consider any convex polygon of $n+1$ vertices. Label the $n+1$ vertices $p_1, p_2, \ldots, p_n, p_{n+1}$, in counterclockwise order. Draw the line segment $\overline{p_1 p_n}$. The polygon of $n+1$ vertices is the union of the convex polygon with n vertices $p_1, p_2, \ldots, p_n$ and the triangle with vertices p_1, p_n, and p_{n+1}. By the inductive hypothesis, the polygon with n vertices $p_1, p_2, \ldots, p_n$ has a sum of interior angles equal to $(n-2)p$. The triangle with vertices p_1, p_n, and p_{n+1} has a sum of interior angles equal to p . Thus the sum of all interior angles is $(n-2)p + p = ((n+1)-2)p$.