Cardinality Problems

1. Using Definition 2', prove that the set of even integers is infinite.

2. Using only Definition 2', show that the set of negative integers is infinite.

3. Using only Definition 2', show that the set $\mathbb{Z}\times\mathbb{Z}$ of ordered pairs of integers is infinite.

4. Given two sets A and B, we say the sets are cardinally equivalent if and only if there exists a one-to-one mapping of A onto B. Denote this by $A \approx B$ and prove that "$\approx$" is an equivalence relation on any set of sets $\mathcal{A}$.

★5. Let A be a set that is infinite according to Definition 1. Define $F = \{ f \mid \text{for some } n \geq 1,\ f : \{1,2,...,n\} \xrightarrow{1-1} A \}$. Now define a partial order "$\prec$" on F as functional extension: for $f, g \in F$, $f \prec g$ if the domain of f is a subset of the domain of g and $f(i) = g(i)$ for all i in the domain of f. Use this partial order and Zorn's Lemma to show that there exists a function $f_\infty : \mathbb{N}^+ \xrightarrow{1-1} A$. Let $A' = f_\infty(A)$ and define $h : A' \xrightarrow{1-1} A'$ by $h(a) = f_\infty(1 + f_\infty^{-1}(a))$ for $a \in A'$. Show that $h(A')$ is a proper subset of A' and then carefully extend h to all of A. Conclude that A is also infinite according to Definition 2.

6. Suppose the set A is uncountably infinite, the set B is countably infinite and $C = A \sim B$. Prove or disprove (with a simple counter example) that C is uncountably infinite.

7. Suppose the set A is uncountably infinite, the set B is countably infinite, and the set C is finite. Let $D = A \cup B \cup C$. Is D finite, countably infinite, or uncountably infinite? Prove your claim.

8. The set of all strings of zeros and ones of infinite length is uncountably infinite. Is the subset of all such strings with a finite number of ones finite, countably infinite, or uncountably infinite? Prove your assertion.

9. Is the set of all strings of zeros and ones of even length finite, countably infinite, or uncountably infinite? Prove your assertion.

10. Consider the set E of all ordered pairs of real numbers of the form (a,b) where $a \leq b$. Is E finite, countably infinite, or uncountably infinite? Prove your claim.

11. Given that the half-closed/half-open interval $[0,1)$ is uncountably infinite, prove that the annulus $A = \{(x,y) \mid x \text{ and } y \text{ are real and } \frac{1}{2} \leq x^2 + y^2 \leq 1\}$ is uncountably infinite.

12. Prove that the set $[0,1]\times[0,1]$ is uncountably infinite.

13. Suppose the set A is non-empty and the set B is uncountably infinite. Prove that the Cartesian product $A \times B$ is uncountably infinite.

14. Prove that for $n \geq 1$, if A_1, A_2, , A_n are finite, then so is $A_1 \times A_2 \times \cdots \times A_n$.

15. Prove that for $n \geq 1$, if A_1, A_2, , A_n are countably infinite, then so is $A_1 \times A_2 \times \cdots \times A_n$.

16. Prove that for $n \geq 1$, if A_1, A_2, , A_n are uncountably infinite, then so is $A_1 \times A_2 \times \cdots \times A_n$.

17. Prove that the set of numbers in the interval [0, 1] having only the digits 0 or 1 in their decimal expansions is uncountably infinite.

> We can show the set A of numbers in the interval [0, 1] having only the digits 0 or 1 in their decimal expansions is uncountably infinite if we construct a one-to-one mapping of the uncountably infinite set [0, 1] into A. To that end, consider the binary expansion $.b_1b_2b_3\ldots$ of any $x \in [0,1]$ (using a terminal sequence of 1's if there is an option between a terminal sequence of 1's or a terminal sequence of 0's). Define $f:[0,1] \to A$ by $f(x) = .b_1b_2b_3\ldots$ interpreted in decimal - that is for $x = \sum_{k=1}^{\infty} b_k 2^{-k}$, $f(x) = \sum_{k=1}^{\infty} b_k 10^{-k}$. Since there are only 0's and 1's in the binary expansion of *x*, there are only 0's and 1's in the decimal expansion of any *f(x)*. We must show this function is one-to-one. If $x \neq y$ then their binary expansions must differ: there exists *i* so that the binary representation of *x* has a zero in position *i* but the binary representation of y has a one in that position (or vice versa). But then the decimal expansions of *f(x)* and *f(y)* must also differ in position *i*, so $f(x) \neq f(y)$ and *f* is one-to-one.

18. Prove that there is no set C such that its power set P(A)is countably infinite. (Hint: You might want to exploit Theorem 11.)

> Let $C_k = \{(i,j) | i, j \in \mathrm{N} \ and \ i, j \leq k\}$, then $\mathrm{N} \times \mathrm{N} = \bigcup_{k \in \mathrm{N}} C_k$. By Theorem 6, $\mathrm{N} \times \mathrm{N}$ is countable. By Theorem 1, N is countable and we define $f : \mathrm{N} \to \mathrm{N} \times \mathrm{N}$ by $f(n) = (n,0)$. For $n, m \in \mathrm{N}, n \neq m \Rightarrow f(n) = (n,0) \neq (m,0) = f(m)$, so f is one to one and thus by Theorem 4, $\mathrm{N} \times \mathrm{N}$ is infinite. We conclude that $\mathrm{N} \times \mathrm{N}$ is countable infinite.

19. Suppose:

(i.) The set *S* of all characters occurring in a computer program is finite.

(ii.) The set of computer programs is the subset of finite strings of characters from S that obey certain syntax rules.

Prove that the set of computer programs is countable.

Let A_k represent the set of all strings of length k from S, for $k = 0, 1, \dots$. Each A_k is finite since S is. The set of all finite strings is $A = \bigcup_{k=0}^{\infty} A_k$ and this is a countable union of countable sets, and thus is countable. The set of all computer programs is a subset of this and thus, either by the corollary to Theorem 3 or by Theorem 8, is countable.

20. Suppose that for any set A there exists no function f mapping A one-to-one onto its power set P(A). Prove that no set has a countably infinite power set.

21. Prove that for no non-empty set A, there exist no function f mapping A one-to-one onto its power set $\mathrm{P}(A)$. (Hint: consider $C = \left\{ a \in A | a \notin f(a) \right\}$.)