Combinatorial Arguments

These are not algebraic arguments. There may be a **tiny** bit of algebra involved but in general the entirety of the argument hangs on combinatorics.

The format for the proof is this:

a. Present a model (i.e. a combinatorial problem). "How many subsets ...?", "how many strings ...?", whatever.

b. Solve the problem with an answer that looks like the left hand side of the equation.

c. Solve the problem with an answer that looks like the right hand side of the equation.

Since the problem has only one solution, the right side must equal the left side.

The way to figure out the model is to stare at both sides and see if one side suggests itself as the count of something. When you see products and powers, you know that suggest independent options. When you see sums, you know that suggests either/or type of cases.

Examples:

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

How I figured it out: I see the 2^n and immediately think that there are 2 options on each of n choices. Like maybe having a bit string of length n since for each position there is the option of having a 0 or a 1. I look at the left hand side and see the summation. That suggest cases - and the cases are indexed by the variable k. Lastly k takes on all values from 0 through n. So how could I bust the bit string problem into such cases? Idea: let k be the number of 1's present in the bit string. There are n positions in the string and $\binom{n}{k}$ ways to select the k positions holding 1's, so this fits.

That was the thinking. Here is the

Proof: Consider a model of strings of length n containing either 0's or 1's. Since for each of the n positions, there are two options - and the options are independent, there are 2^n such strings. This agrees with the right hand side. To argue the left hand side, let k be the number of 1's in such a string. The number of 1's varies from 0 through n and for a fixed number k of 1's there are $\binom{n}{k}$ ways to position the 1's. Thus the summation from $\sum_{k=0}^{n} \binom{n}{k}$ must equal 2^n.

2. $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Here's my thinking: The left hand side makes the model pretty obvious - we are choosing k+1 objects from n+1 without repetition and without concern for order. The "+1" 's hanging around suggest we might want to break the set of n+1 into two subsets: one of size n and a singleton. But that's just a suggestion and we should stare at the other side to see what it looks like. The + sign should say to us "cases" and thus there should be two cases: one in which k is being chosen from n and another case where k+1 is being taken from n. OK, we draw a little picture with a big circle representing a set of n objects and a little circle beside it representing just one object. We see that to choose k+1 objects from the union we must **EITHER** choose the little guy **OR** not. If we choose the little guy, then we have k to choose from the big set (thus we get $\binom{n}{k}$ options). On the other hand, if we do not choose the little guy, we must choose all k+1 from the big set of n (thus $\binom{n}{k+1}$).

That's the thinking - and does not go into your

Proof: Let A be a set of size n and b be an element not contained in A. Let $C = A \cup \{b\}$. How many subsets of C have exactly k+1 elements? Since C has cardinality n+1, there are $\binom{n+1}{k+1}$ such subsets. This agrees with the left hand side. For the right hand side, consider that a subset must either contain b or not (and not both). If the subset contains b, then there are k remaining elements of the subset to be selected from A. There are $\binom{n}{k}$ ways to do that. If the subset does not contain b, then all k+1 elements must be selected from A. There are $\binom{n}{k+1}$ ways to do that, thus there are $\binom{n}{k} + \binom{n}{k+1}$ total ways and this must equal $\binom{n+1}{k+1}$.