Matrix Norms

Lemma: *Given any vector norm* $\|\cdot\|$ *and any non-zero vector* x*,* $\frac{1}{\|x\|}x$ *is a unit vector. That is*

$$\left\| \frac{1}{\|x\|} x \right\| = 1.$$

Theorem 1.: *Given vector norms* $\|\cdot\|_\alpha$ *and* $\|\cdot\|_\gamma$ *defined for* $m-$ *and* $n-$*dimensional real vector spaces, respectively, , define the real values function on the space of* $m \times n$ *real matrices,*

$$N(A) = \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\}$$

for all $m \times n$ *real matrices* A*. This function is a norm.*

Proof:

Since $\|\cdot\|_\gamma$ is a vector norm, we have that for any x, $\|Ax\|_\alpha \geq 0$. Thus

$$N(A) = \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\} \geq 0.$$

If A is the zero matrix, then for any x, $\|Ax\|_\alpha = 0$, so $\max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\} = 0$. If A is not the zero matrix, then it must have a non-zero element - call it $a_{i,j}$. Letting e_j be the j^{th} coordinate vector, then with $\bar{e}_j = \frac{1}{\|e_j\|_\gamma} e_j$, we have $A\bar{e}_j \neq 0$ so $\|A\bar{e}_j\|_\alpha > 0$, and $\|\bar{e}_j\|_\gamma = 1$ so $N(A) = \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\} \geq \|A\bar{e}_j\|_\alpha > 0.$ For any real value a,

$$\begin{aligned} N(aA) &= \max_{\|x\|_\gamma = 1} \{\|aAx\|_\alpha\} \\ &= \max_{\|x\|_\gamma = 1} \{|a|\,\|Ax\|_\alpha\} \\ &= |a| \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\} \\ &= |a|\, N(A). \end{aligned}$$

Lastly,

$$\begin{aligned} N(A+B) &= \max_{\|x\|_\gamma = 1} \{\|(A+B)x\|_\alpha\} \\ &= \max_{\|x\|_\gamma = 1} \{\|Ax + Bx\|_\alpha\} \\ &\leq \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha + \|Bx\|_\alpha\} \\ &\leq \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\} + \max_{\|x\|_\gamma = 1} \{\|Ax\|_\alpha\} \\ &\leq N(A) + N(B). \end{aligned}$$

Thus, N is a norm. ▪

Theorem 2 gives an equivalent definition of the function N.

Theorem 2.: *The function N defined in Theorem 1 satisfies,*

$$N(A) = \sup_{x \neq 0}\{\frac{\|Ax\|_\alpha}{\|x\|_\gamma}\}.$$

for all $m \times n$ real matrices A.

Proof: For any $\gamma-$ norm unit vector x,

$$\begin{aligned}\|Ax\|_\alpha &= \frac{\|Ax\|_\alpha}{\|x\|_\gamma} \\ &\leq \sup_{x \neq 0}\{\frac{\|Ax\|_\alpha}{\|x\|_\gamma}\}\end{aligned}.$$

Since this holds for any $\gamma-$ norm unit vector x,

$$N(A) = \max_{\|x\|_\gamma = 1}\{\|Ax\|_\alpha\} \leq \sup_{x \neq 0}\{\frac{\|Ax\|_\alpha}{\|x\|_\gamma}\}$$

Also for any $\gamma-$ norm unit vector x,

$$\|Ax\|_\alpha \leq \max_{\|x\|_\gamma = 1}\{\|Ax\|_\alpha\} = N(A).$$

We have

$$N(A) \leq \sup_{x \neq 0}\{\frac{\|Ax\|_\alpha}{\|x\|_\gamma}\} \leq N(A),$$

so the theorem is established. ▪

Given vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\gamma$ defined for $m-$ and $n-$dimensional real vector spaces, and matrix norm $\|\cdot\|_\beta$ defined for $m \times n$ real matrices, we say *the matrix norm is consistent with the vector norms* if for all $m \times n$ real matrices A and all real $n-$vectors x, the real $m-$vector Ax satisfies

$$\|Ax\|_\alpha \leq \|A \cdot\|_\beta \|x\|_\gamma.$$

Theorem 3 establishes that the norm defined in Theorem 1 is the minimal consistent matrix norm. As such, we term it *the matrix norm subordinate to the vector norms* $\|\cdot\|_\alpha$ *and* $\|\cdot\|_\gamma$.

Theorem 3.: *Given vector norms* $\|\cdot\|_\alpha$ *and* $\|\cdot\|_\gamma$ *defined for* $m-$ *and* $n-$*dimensional real vector spaces, and matrix norm* $\|\cdot\|_\beta$ *defined for* $m \times n$ *real matrices consistent with the vector norms,*

$$N(A) \le \|A\|_\beta ,$$

for all $m \times n$ *real matrices* A.

Proof:

For any non-zero, real $n-$vectors x, $\|Ax\|_\alpha \le \|A \cdot\|_\beta \|x\|_\gamma$, so

$$\frac{\|Ax\|_\alpha}{\|x\|_\gamma} \le \|A \cdot\|_\beta$$

The right hand side is independent of x, so

$$\sup_{x \ne 0}\{\frac{\|Ax\|_\alpha}{\|x\|_\gamma}\} \le \|A \cdot\|_\beta ,$$

but according to Theorem 2, $N(A) = \sup_{x \ne 0}\{\frac{\|Ax\|_\alpha}{\|x\|_\gamma}\}$. ▪

As an example, we show that with both vector norms taken as the ∞ norm (on $m-$ and $n-$dimensional real vector spaces, respectively) the subordinate vector norm is equal to the maximal rows sum of absolute values.

Theorem 4: *Let* A *be an* $m \times n$ *real matrix. Then*

$$\max_{i=1,\dots,m}\{\sum_{j=1}^{n} |a_{i,j}|\} = \max_{\|x\|_\infty = 1}\{\|Ax\|_\infty\} .$$

Thus, the matrix norm subordinate to the $\|\cdot\|_\infty$ *vector norm is*

$$\|A\|_\infty = \max_{i=1,\dots,m}\{\sum_{j=1}^{n} |a_{i,j}|\} .$$

Proof:

Consider any ∞ norm unit vector x. Since each of its components satisfies $|x_j| \le 1$, we have for $i = 1,\dots,m$

$$\left|\sum_{j=1}^{n} a_{i,j}x_j\right| \le \sum_{j=1}^{n} |a_{i,j}x_j|$$

$$\le \sum_{j=1}^{n} |a_{i,j}||x_j|$$

$$\le \sum_{j=1}^{n} |a_{i,j}| .$$

Thus by taking the maximum over i,

$$|\sum_{j=1}^{n} a_{i,j} x_j| \le \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{i,j}|,$$

holds for any ∞ norm unit vector x, thus

$$\|Ax\|_\infty = \max_{i=1,\dots,m} |\sum_{j=1}^{n} a_{i,j} x_j| \le \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{i,j}|,$$

and by taking the maximum over all such vectors, we have

$$\max_{\|x\|_\infty=1} \{\|Ax\|_\infty\} \le \max_{i=1,\dots,m} \{\sum_{j=1}^{n} |a_{i,j}|\}.$$

For a fixed matrix A, let k be the index of a row with maximal row sum - that is,

$$\sum_{j=1}^{n} |a_{k,j}| = \max_{i=1,\dots,m} \{\sum_{j=1}^{n} |a_{i,j}|\}.$$

Now define a vector x as

$$x_j = \begin{cases} 1 & if\ a_{k,j} \ge 0 \\ -1 & if\ a_{k,j} < 0 \end{cases}.$$

Notice that for $j = 1,\dots,n$, $a_{k,j} x_j = |a_{k,j} x_j| = |a_{k,j}|$, so

$$\begin{aligned} \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{i,j}| &= \sum_{j=1}^{n} |a_{k,j}| \\ &= \sum_{j=1}^{n} a_{k,j} x_j \\ &= \left| \sum_{j=1}^{n} a_{k,j} x_j \right| \\ &\le \max_{i=1,\dots,m} \left| \sum_{j=1}^{n} a_{i,j} x_j \right| \\ &= \|Ax\|_\infty \end{aligned}$$

This hold for a particular ∞ norm unit vector x, so by taking a maximum

$$\max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{i,j}| \le \max_{\|x\|_\infty=1} \|Ax\|_\infty.$$

We have now

$$\max_{\|x\|_\infty=1} \|Ax\|_\infty \le \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{i,j}| \le \max_{\|x\|_\infty=1} \|Ax\|_\infty,$$

so

$$\|Ax\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{i,j}|. \blacksquare$$