Exam Review Problem Solutions

1. Finding inverses:

a. Determine the inverse of $A=\begin{bmatrix}2 & 3 & 1\\ 4 & 7 & 5\\ 0 & -2 & 2\end{bmatrix}$.

If in elimination we carry along the three columns of the identity as right hand sides, we have

$$\begin{bmatrix}2 & 3 & 1 & 1 & 0 & 0\\ 4 & 7 & 5 & 0 & 1 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\end{bmatrix}\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 2 & 3 & 1 & 1 & 0 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\end{bmatrix}\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & -1/2 & -3/2 & 1 & -1/2 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\end{bmatrix}$$

$$\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\\ 0 & -1/2 & -3/2 & 1 & -1/2 & 0\end{bmatrix}\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\\ 0 & 0 & -2 & 1 & -1/2 & -1/4\end{bmatrix}$$

Then back solving for the three columns of the inverse:

$$\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\\ 0 & 0 & -2 & 1 & -1/2 & -1/4\end{bmatrix}\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & -2 & 2 & 0 & 0 & 1\\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8\end{bmatrix}\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & -2 & 0 & 1 & -1/2 & 3/4\\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8\end{bmatrix}$$

$$\to\begin{bmatrix}4 & 7 & 5 & 0 & 1 & 0\\ 0 & 1 & 0 & -1/2 & 1/4 & -3/8\\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8\end{bmatrix}\to\begin{bmatrix}4 & 0 & 0 & 6 & -2 & 2\\ 0 & 1 & 0 & -1/2 & 1/4 & -3/8\\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8\end{bmatrix}$$

$$\to\begin{bmatrix}1 & 0 & 0 & 3/2 & -1/2 & 1/2\\ 0 & 1 & 0 & -1/2 & 1/4 & -3/8\\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8\end{bmatrix}$$

so the inverse is $\begin{bmatrix}3/2 & -1/2 & 1/2\\ -1/2 & 1/4 & -3/8\\ -1/2 & 1/4 & 1/8\end{bmatrix}$

So $A^{-1}=\begin{bmatrix}3/2 & -1/2 & 1/2\\ -1/2 & 1/4 & -3/8\\ -1/2 & 1/4 & 1/8\end{bmatrix}$.

b. Determine the inverse of the elementary row operation $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$. (You can do all the arithmetic or you can ask yourself "How can I undo what B does?.)

Since B subtracts ½ of the second component from the third, B^{-1} must add ½ of the second component to the third. $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$.

c. Determine the inverse of the rotation $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3/4} \\ 0 & \sqrt{3/4} & 1/2 \end{bmatrix}$. (Again, you can do the arithmetic or you can ask yourself "How can I undo what R does?.)

Since R rotates the second and third through some angle, R^{-1} must rotate the second and third through some the negative of the angle. $R^{-1} = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3/4} \\ 0 & -\sqrt{3/4} & 1/2 \end{bmatrix}$.

2. Finding a null space and a determinant of a 3 by 3 matrix:

a. Show that $C = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & \mathbf{1} \\ 0 & -2 & 2 \end{bmatrix}$ is singular by finding a non-zero vector in its null space. (You might save yourself some work by noticing the slight difference between C and A from Problem 1a.)

We get

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 7 & 1 \\ 2 & 3 & 1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 7 & 1 \\ 0 & -1/2 & 1/2 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 7 & 1 \\ 0 & -2 & 2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \to \begin{bmatrix} 4 & 7 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so since $\begin{bmatrix} 4 & 7 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ the vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is in the null space of C.

b. Confirm that the determinant of C is zero.

$$\det(C) = 2\cdot 7\cdot 2 + 3\cdot 1\cdot 0 + 1\cdot 4\cdot(-2) - 1\cdot 7\cdot 0 - 2\cdot 1\cdot(-2) - 2\cdot 4\cdot 3$$
$$= 28 + 0 - 8 - 0 + 4 - 24 = 0.$$

3. Finding eigenvectors:

Find the eigenvalues and eigenvectors of $D = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 7 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

Since this is upper triangular, the eigenvalues are 2,7, and 2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3/5 \\ 1 \\ 0 \end{bmatrix}$. The null space of $D - 2I = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ contains only the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so it is the only eigenvector for the eigenvalue 2.

4. Vector Spaces:

Identify which of the following satisfy the definition for vector spaces. For each case either mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

	Closed under addition	Closed under scalar multiplication
a. The set of 4 by 2 matrices.	___Yes___	___Yes___
b. The set of 3 by 3 singular matrices.	___No___	___Yes___

The 3 by 3 matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are singular but

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is nonsingular.

c. The set of 3 by 3 nonsingular matrices.	___No___	___No___

The 3 by 3 identity and its negative are non-singular but the zero matrix $= I + (-I)$ is singular.

The 3 by 3 identity is non-singular but the zero matrix $= 0 \cdot I$ is singular.

d. The set of 3-vectors that are perpendicular to **both** [1, 2, 3] **and** [-1, 0, 1].	___Yes___	___Yes___
e. The set of 3-vectors that are perpendicular to **either** [1, 2, 3] **or** [-1, 0, 1].	___No___	___Yes___

[2,-1,0] is perpendicular to [1, 2, 3] and [0,1,0] is perpendicular to [-1, 0, 1] but [2, 0, 0]=[2,-1,0] + [0,1,0] is perpendicular to neither.

5. Column Space:

a. How many linearly independent vectors are in the column space of matrix $A=\begin{bmatrix}2 & 3 & 1\\ 4 & 7 & 5\\ 0 & -2 & 2\end{bmatrix}$?

As was found in Problem 1, A has an inverse so the column space fills $\mathbb{R}^3$. There are three linearly independent vectors.

b. How many linearly independent vectors are in the column space of matrix $C=\begin{bmatrix}2 & 3 & 1\\ 4 & 7 & \mathbf{1}\\ 0 & -2 & 2\end{bmatrix}$?

As found in Problem 2, C can be reduced through elimination to $\begin{bmatrix}4 & 7 & 1\\ 0 & -2 & 2\\ 0 & 0 & 0\end{bmatrix}$. The null space has dimension one so the column space has dimension two.

c. For any matrix E and vector b, describe a solution of $Ex=b$ in terms of the column space of E.

If $Ex=b$ then b is in the column space of matrix E .

6. Proofs:

a. Prove that if y is an eigenvector of F then y is also an eigenvector of $6F^3-2F+3I$.

We have for some λ, $Fy=\lambda y$ so $-2Fy=-2\lambda y, 3Iy=3y$, and
$6F^3y=6F^2(Fy)=6F^2(\lambda y)=6F(\lambda Fy)=6F(\lambda^2 y)=6\lambda^2 Fy=6\lambda^3 y$,
and $(6F^3-2F+3I)y=(6\lambda^3-2\lambda+3)y$ so y is also an eigenvector of $6F^3-2F+3I$.

b. Prove that if an inverse matrix A^{-1} exists so that $A^{-1}A=I$ then the null space of A contains only the zero vector.

Suppose x is in the null space of A, then $x=Ix=A^{-1}Ax=A^{-1}0=0$.

c. Prove for any n by n matrices G and H (where H is non-singular) that if λ is an eigenvalue of G with corresponding eigenvector v then λ is an eigenvalue of HGH^{-1} with corresponding eigenvector Hv.

We have $(HGH^{-1})(Hv)=HG(H^{-1}H)v=HGv=H(\lambda v)=\lambda Hv$. Thus, λ is an eigenvalue of HGH^{-1} with corresponding eigenvector Hv.

7. Gram-Schmidt and Least Squares:

a. Using the Gram-Schmidt Algorithm transform the matrix $M = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & 5 \\ 4 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ into a product

$M = QR$, where $Q^T Q = I$ and R is upper triangular.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{21}} & \frac{8}{\sqrt{105}} & \frac{46}{\sqrt{6230}} \\ 0 & \frac{-6}{\sqrt{105}} & \frac{179}{3\sqrt{6230}} \\ \frac{4}{\sqrt{21}} & \frac{-1}{\sqrt{105}} & \frac{-56}{3\sqrt{6230}} \\ \frac{2}{\sqrt{21}} & \frac{-2}{\sqrt{105}} & \frac{43}{3\sqrt{6230}} \end{bmatrix}, R = \begin{bmatrix} \sqrt{21} & \frac{7}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \\ 0 & \frac{35}{\sqrt{105}} & \frac{2}{\sqrt{105}} \\ 0 & 0 & \frac{534}{\sqrt{6230}} \end{bmatrix}$$

b. Using this, for $b = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$, determine x that minimizes $\|Mx - b\|$. .

We get $Q^T b = \begin{bmatrix} \frac{-4}{\sqrt{21}} \\ \frac{2\sqrt{5}}{\sqrt{21}} \\ \frac{61\sqrt{10}}{3\sqrt{623}} \end{bmatrix}$ and then solve $Rx == \begin{bmatrix} \sqrt{21} & \frac{7}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \\ 0 & \frac{35}{\sqrt{105}} & \frac{2}{\sqrt{105}} \\ 0 & 0 & \frac{534}{\sqrt{6230}} \end{bmatrix} x = \begin{bmatrix} \frac{-4}{\sqrt{21}} \\ \frac{2\sqrt{5}}{\sqrt{21}} \\ \frac{61\sqrt{10}}{3\sqrt{623}} \end{bmatrix}$ to get

$$x = \begin{bmatrix} \frac{-194}{801} \\ \frac{1480}{5607} \\ \frac{305}{801} \end{bmatrix}.$$