M 340L – CS
Homework Set 3 Solutions

1. a. Perform the computation of $(1.3579-(975310.23/34567.89))\times.00023456$ entirely in 3 decimal digit rounding floating point.

$1.3579 \rightarrow 1.36$
$975310.23 \rightarrow 975000$
$34567.89 \rightarrow 34600$
$.00023456 \rightarrow .000235$

$975000/34600 = 28.179190751445088... \rightarrow 28.2$
$1.36-28.2 = \text{-}26.84 \rightarrow \text{-}26.8$
$\text{-}26.8\times.000235\text{=-}0.006298 \rightarrow \text{-}.00630$

b. What is the result of the same computation in exact arithmetic?

$$(1.3579-(975310.23/34567.89))\times.00023456 = \text{-}0.006299446759497...$$

c. What is the error?

$$-.00630 \text{ - } (\text{-}0.006299446759497...) = \text{-}.0000055324...$$

d. What is the (absolute) relative error?

$$\left|\frac{.00000055324...}{\text{-}0.006299446759497...}\right| \text{=}.00008782366512143027...,$$

2. **Backward Error**: The 3 decimal digit rounding floating point result of computing $(36.258-31.9876)+537.862$ is 542 . Find operands *a, b,* and *c* close to $36.258, 31.9876, \text{and } 537.862$ (relative to the largest of them), respectively, so that the exact computation of *(a-b)+ c* is 542. (Notice we do not compare the computed result 542 to the exact result 542.1324... .)

Let $a = 36.2138, b = 32.0317, \text{and } c = 537.8179$, then *(a-b)+ c* is 542 and the changes to $36.258, 31.9876, \text{and } 537.862$ are about .044.

3. Determine if the system has a nontrivial solution. (One row operation is sufficient to determine the answer.)

$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 0 \\ -3x_1 - 4x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 5 & -3 & 2 & 0 \\ -3 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 5 & -3 & 2 & 0 \\ 0 & -29/5 & 16/5 & 0 \end{bmatrix}.$$ x_3 is free so there is a nontrivial solution.

4. Write the solution set of the given homogeneous system in parametric vector form. (Follow the method of Examples 1 and 2 in Lay 1.5)

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -x_1 + x_2 \qquad &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, $x_2 = x_3, x_1 = x_3$. In parametric vector form this is $x = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

In problems 5 and 6, describe all solutions of $Ax = 0$ in parametric vector form, where A is row equivalent to the given matrix.

5.

$$\begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

$\begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -7 \\ 0 & 1 & 2 & -4 \end{bmatrix}$. x_4 is free and x_3 is free, $x_2 = -2x_3 + 4x_4$, $x_1 = 2x_3 + 7x_4$. In parametric vector form this is $x = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$.

6.

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 0 & 29 & -36 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 0 & 29 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$x_6 = 0, x_5$ is free, $x_4 = -4x_5, x_3$ is free, x_2 is free, and $x_1 = 2x_2 - 3x_3 - 29x_5$. In parametric vector form this is $x = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$.

7. Mark each statement True or False then justify your answer.

a. A homogeneous system of equations can be inconsistent.

False. Since the zero vector is always a solution, a homogeneous system of equations can never be inconsistent.

b. If x is a nontrivial solution of $Ax = 0$, then every entry in x is nonzero.

False. A nontrivial solution need only have at least one non-zero component.

c. The effect of adding vector p to a vector x is to move the vector x in a direction parallel to p.

True. The vertices $0, p, x + p, x$ form a parallelogram so the effect of adding p to x to get $x + p$ is a line parallel to the line from zero to p.

d. The equation $Ax = b$ is homogeneous if the zero vector is a solution.

True. Since $A0 = b = 0$, the equation $Ax = b$ is also $Ax = 0$ and thus is homogeneous.

8. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

9. Given $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection.

Since twice the first column plus three times the third column is zero, $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

10. a. Use the **original** version of the Gaussian Elimination Algorithm to solve $\begin{bmatrix} .001 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Do this with exact arithmetic.

To solve $\begin{bmatrix} .001 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ exactly we have for matrix A in storage

$$\left[\begin{array}{cc|c} .001 & -1 & 1 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} .001 & -1 & 1 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} .001 & -1 & 1 \\ 1 & 1002 & -1000 \end{array}\right]$$

The solution is $x_2 = \dfrac{-1000}{1002} = -.99803992...$,

$$x_1 = \frac{1-(-1)(\frac{-1000}{1002})}{.001} = \frac{2000}{1002} = 1.996007984....$$

b. Now employ the **same** algorithm on the same problem but simulate a three decimal digit floating point environment (thus $2+1000$ is computed as 1000).

To solve $\begin{bmatrix} .001 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we would have for matrix A in storage

$\left[\begin{array}{cc|c} .001 & -1 & 1 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1000 & -1000 \end{array}\right]$ (since $2+1000$ is computed as 1000). The computed solution is $x_2 = \dfrac{-1000}{1000} = -1$, and $x_1 = \dfrac{1-(-1)(-1)}{1} = 0.$

c. Finally, use the **same** algorithm and do the work again in a three decimal digit floating point environment (thus $-1-.002$ is computed as -1) but swap the two rows so you are solving $\begin{bmatrix} 1 & 2 \\ .001 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

To solve $\begin{bmatrix} 1 & 2 \\ .001 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with the original algorithm we would have for matrix A in storage $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ .001 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ .001 & -1 & 1 \end{array}\right]$ (since $-1-.002$ is computed as -1).The computed solution is $x_2 = \dfrac{1}{-1} = -1$, and $x_1 = \dfrac{0-2(-1)}{1} = 2.$