M 340L – CS
Homework Set 7 Solutions

1. Let $u=\begin{bmatrix}-1\\2\end{bmatrix}, v=\begin{bmatrix}4\\6\end{bmatrix}, w=\begin{bmatrix}3\\-1\\-5\end{bmatrix}, x=\begin{bmatrix}6\\-2\\3\end{bmatrix}$. Compute:

a. $w\cdot w=3^2+(-1)^2+(-5)^2=35.$

b. $x\cdot w=6\cdot 3+(-2)\cdot(-1)+3\cdot(-5)=5.$

c. $\dfrac{x\cdot w}{w\cdot w}=\dfrac{5}{35}=\dfrac{1}{7}.$

d. $\dfrac{1}{u\cdot u}u=\dfrac{1}{(-1)^2+2^2}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}-1/5\\2/5\end{bmatrix}.$

e. $\|x\|=\sqrt{6^2+(-2)^2+3^2}=\sqrt{49}=7.$

2. Find a unit vector in the direction $\begin{bmatrix}-6\\4\\-3\end{bmatrix}$.

To normalize $\begin{bmatrix}-6\\4\\-3\end{bmatrix}$, we have $\dfrac{1}{\sqrt{(-6)^2+4^2+(-3)^2}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\dfrac{1}{\sqrt{61}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\begin{bmatrix}-6/\sqrt{61}\\4/\sqrt{61}\\-3/\sqrt{61}\end{bmatrix}.$

3. Find the distance between $u=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $z=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$.

The distance between $u=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $z=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$ is

$$\|u-z\|=\left\|\begin{bmatrix}0\\-5\\2\end{bmatrix}-\begin{bmatrix}-4\\-1\\8\end{bmatrix}\right\|=\left\|\begin{bmatrix}4\\-4\\-6\end{bmatrix}\right\|=\sqrt{4^2+(-4)^2+(-6)^2}=\sqrt{68}.$$

4. Answer true or false to the following. If false offer a simple counterexample.

a. $u \cdot v - v \cdot u = 0$.

True. We have commutatively of the dot product.

b. For any scalar c, $\|cv\| = c\|v\|$.

False. $\|-1[1]\| = \|[-1]\| = 1 \neq -1 = -1\|[1]\|.$.

c. If $\|u\|^2 + \|v\|^2 = \|u+v\|^2$, then u and v are orthogonal.

True. We have $\|u+v\|^2 = \|u\|^2 + 2u \cdot v + \|v\|^2 = \|u\|^2 + \|v\|^2$, so $u \cdot v = 0$ and u and v are orthogonal.

d. For an $m \times n$ matrix A and $1 \leq i \leq m$, if x in the null space of A then x is orthogonal to $A_{i.}$, the i^{th} row of A.

True. If x in the null space of A then $Ax = 0$ but the i^{th} component of Ax is $A_{i.} \cdot x$, which is zero.

5. Verify the parallelogram law for vectors u and v in $\mathbb{R}^n$: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

We have $\|u+v\|^2 + \|u-v\|^2 = \|u\|^2 + 2u \cdot v + \|v\|^2 + \|u\|^2 - 2u \cdot v + \|v\|^2 = 2\|u\|^2 + 2\|v\|^2$.

6. Given vectors u and v in $\mathbb{R}^n$, consider vectors of the form $v + \alpha u$, for all scalars α.

a. Determine α so that $v + \alpha u$, is orthogonal to u.

For orthogonality, we want $0 = u \cdot (v + \alpha u) = u \cdot v + \alpha u \cdot u = u \cdot v + \alpha\|u\|^2$. Thus,

$\alpha = -\dfrac{u \cdot v}{\|u\|^2}$, if $\|u\|^2 \neq 0$.

b. Under what circumstance is there no such α?

The quantity $\alpha = -\dfrac{u \cdot v}{\|u\|^2}$ is not defined if $\|u\|^2 = 0$, (that is, if $u = 0$). However, in that case the vector $v + \alpha u$ is orthogonal to u for any α.