M 340L CS
Course Outline

Lecture 1:

1. Syllabus and course procedures
2. What is in this course?

Lecture 2:

1. Linear system of equations
2. Gaussian (Liu Hui) Elimination
3. Matrix Form of equations – “augmented form”
4. Elementary Row Operations and equivalent solutions
   a. SAXPY
   b. Swap
   c. Scale
5. Existence/Uniqueness
6. Row reduction to Triangular Form
7. Back Substitution

Lecture 3:

1. Examples:
2. Echelon Form
   a. Leading non-zero entry in any row has only zeroes beneath it.
   b. Indices of leading non-zero entries are strictly increasing with row number.
3. Pivot/Pivot Position/Pivot Column
4. Vectors in $\mathbb{R}^n$
   a. sum/differences
   b. scalar multiples
5. Geometric interpretation of sums and multiples
6. Algebraic properties of sums and scalar multiples
   For all u; v; w in $\mathbb{R}^n$ and all scalars c and d :
   $u + v = v + u$
   $(u + v) + w = u + (v + w)$
   $u + 0 = u$
   $u + (-u) = u + (-1)u = 0$
   $c(u + v) = cu + cv$
   $(c + d)u = cu + du$
   $c(du) = (cd)u$
   $1u = u$
7. Linear combinations of vectors
8. Span of a set of vectors

Lecture 4:

1. Free variables/basic variables
2. Existence and Uniqueness Theorem
3. Matrix-vector products as linear combination of columns
4. Systems of linear equations in matrix form
5. First Equivalence Theorem

    Let A be an m x n matrix. Then the following statements are logically equivalent.

    a. For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

    b. Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

    c. The columns of A span $\mathbb{R}^m$.

    d. A has a pivot position in every row.

6. Matrix-vector product properties
    a. $A(u+v) = Au + Av$
    b. $A(cu) = c(Au)$
7. Homogeneous linear systems (trivial and non-trivial solutions)
8. Parametric form of solutions

Lecture 5:

1. The original Gaussian algorithm

Lecture 6:

0. Homogeneous systems (more)
1. The original Gaussian algorithm (again)
2. Error Properties
3. Floating Point Numbers
4. Floating Point Arithmetic
5. Cancellation Error
6. Forward and Backward Error

Lecture 7:

1. Matrix Addition, Multiplication, and Powers
2. Properties:
    a. $A+B = B+A$
    b. $(A+B)+C = A+(B+C)$
    c. $A+0= A$
    d. $r(A+B)= rA+rB$
    e. $(r+s)A = rA+sA$
    f. $(sA) = r(sA)$
    g. $A(BC) = (AB)C$
    h. $A(B+C) = AB+AC$
    i. $(A+B)C = AC+BC$
    j. $r(AB) = (rA)B = A(rB)$
    k. $I_m A= A= AI_n$
3. Non-Properties
    a. $AB = BA$
    b. $AB = AC$ implies $B = C$
    c. $AB = 0$ implies $A = 0$ or $B = 0$
4. Matrix Transposes
    a. $(A^T)^T= A$
    b. $(A+B)^T = A^T+B^T$
    c. $(rA)^T = rA^T$
    d. $(AB)^T = B^TA^T$
5. Matrix Inverses
    a. Definition$A^{-1}A= I$ and $AA^{-1}= I$
    b. If A is invertible, the unique solution of $Ax = b$ is $A^{-1}b$

Lecture 8:

1. Partial Pivoting Algorithm
2. Error Properties
3. Operation Count
4. Separate operations on right hand side

Lecture 9

1. Architectural considerations for Gaussian Elimination

Midterm Examination 1

Lecture 10:

1. Matrix Addition, Multiplication, and Powers
2. Properties:
    a. $A+B = B+A$
    b. $(A+B)+C = A+(B+C)$
    c. $A+0= A$
    d. $r(A+B)= rA+rB$
    e. $(r+s)A = rA+sA$
    f. $(sA) = r(sA)$
    g. $A(BC) = (AB)C$
    h. $A(B+C) = AB+AC$
    i. $(A+B)C = AC+BC$
    j. $r(AB) = (rA)B = A(rB)$
    k. $I_mA= A= AI_n$
3. Non-Properties
    a. $AB = BA$
    b. $AB = AC$ implies $B = C$
    c. $AB = 0$ implies $A = 0$ or $B = 0$
4. Matrix Transposes
    a. $(A^T)^T = A$
    b. $(A+B)^T = A^T+B^T$
    c. $(rA)^T = rA^T$
    d. $(AB)^T = B^TA^T$
5. Matrix Inverses
    a. Definition$A^{-1}A= I$ and $AA^{-1}= I$
    b. If A is invertible, the unique solution of $Ax = b$ is $A^{-1}b$
6. Inverse Properties
    a. $(A^{-1})^{-1} = A$
    b. $(AB)^{-1} = B^{-1}A^{-1}$
    c. $(A^T)^{-1} = (A^{-1})^T$
7. Algorithms for finding Inverses

Lecture 11:

1. Elementary Operations as Matrices
    a. Swap
    b. Scale
    c. SAXPY
2. Inverses of Elementary Operations
3. Linear Transformations
    a. Properties
    b. Matrix Representation

Lecture 12:

1. Theorems
   a. T: $R^n \rightarrow R^m$ maps R^n onto R^m if and only if the columns of A span R^m
   b. T: $R^n \rightarrow R^m$ maps R^n one-to-one into R^m if and only if the columns of A are linearly independent
2. The Invertible Matrix Theorem
   Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
   a. A is an invertible matrix.
   b. A is row equivalent to the n x n identity matrix.
   c. A has n pivot positions.
   d. The equation Ax = 0 has only the trivial solution.
   e. The columns of A form a linearly independent set.
   f. The linear transformation $x \rightarrow Ax$ is one-to-one.
   g. The equation Ax = b has at least one solution for each b in R^n.
   h. The columns of A span R^n.
   i. The linear transformation $x \rightarrow Ax$ maps R^n onto R^n.
   j. There is an n x n matrix C such that CA = I .
   k. There is an n x n matrix D such that AD = I .
   l. A^T is an invertible matrix.
3. Block Products

Lecture 13

1. Inner Products
2. Properties
a. $u \cdot v = v \cdot u$
b. $(u+v) \cdot w = u \cdot w + v \cdot w$
c. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
d. $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only $u = 0$
3. Orthogonality
4. Norms and distances

Lecture 14

0. Least Squares Motivation
1. Vector Space (subspace)
2. Basis for vector space
3. (Mutually) Orthogonal/Orthonormal Sets
4. Orthogonal/Orthonormal Basis
5. Expression of vector in orthogonal/orthonormal set

Lecture 15

1. Pythagorean Theorem
2. Orthogonal Projection
   $[\alpha, v'] =$ **project** $[u, v]$
   Inputs vectors u and v, computes and returns $\alpha = u \cdot v / u \cdot u$ and $v' = v - \alpha u.$
3. Orthogonal Matrices
4. Properties of Orthogonal Matrices
   a. $\|x\| = \|Ux\|$
   b. $x \cdot y = (Ux) \cdot (Uy)$
5. Orthogonality Property for Least Squares Problems (begun)

Lecture 16

1. Orthogonality Property for Least Squares Problems
2. Normal Equations
3. Using Gram-Schmidt Orthogonalization to solve Least Squares Problems

Lecture 17

1. Relation of output of Gram-Schmidt Orthogonalization and output of Gram-Schmidt Orthonormalization
2. Gram-Schmidt Orthonormalization Algorithm
   a. Complexity: approximately mn^2 floating point multiplications and mn^2 floating point additions (subtractions)

Lecture 18

1. Application of Least Squares for noise removal
2. Vector Spaces
3. Introduction to (valueless) Determinants

Lecture 19

1. Introduction to (valueless) Determinants
2. MATLAB: a computing environment for scientific computing

Midterm Examination 2

Lecture 20

1. Introduction to Eigenvalues
2. Motivation
3. Characteristic Equation
4. Galois Result

Lecture 21

1. Zero Eigenvalues
2. Number of Eigenvectors
3. Similarity
4. Diagonalization

Lecture 22

1. Determinants of triangular matrices
2. Diagonalization (more)

Lecture 23

1. Power iteration
2. Shifted power iteration
3. Inverse shifted power iteration
4. Powers of matrices

Lecture 24

1. QR iteration
    a. Unshifted
    b. Corner shift
    c. Double shift
2. Deflation

Lecture 25

1. Plane Rotation
2. Hessenberg Form
3. Markov Chains
4. Google Page Rank

Final Examination