M 340L – CS
Homework Set 1

Solve each system in Problems 1–6 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in Lay, Section 1.1.

1.

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}, x_2 = \frac{9}{3} = 3, x_1 = \frac{7 - 5\cdot 3}{1} = -8.$$

2.

$$\begin{aligned} 3x_1 + 6x_2 &= -3 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$$

$$\begin{bmatrix} 3 & 6 & -3 \\ 5 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 3 & 6 & -3 \\ 0 & -3 & 15 \end{bmatrix}, x_2 = \frac{15}{-3} = -5, x_1 = \frac{-3 - 6\cdot(-5)}{3} = 9.$$

3.

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -4 \end{bmatrix},$$

$0x_3 = -4$

No solution.

4.

$$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -9 & 15 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

$0x_3 = 5$

No solution.

5.

$$\begin{aligned} x_1 - 6x_2 &= 5 \\ x_2 - 4x_3 + x_4 &= 0 \\ -x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\ -x_2 + 5x_3 + 4x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ -1 & 6 & 1 & 5 & 3 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix},$$

$0x_4 = 8$

6.

$$\begin{aligned} -4x_4 &= -10 \\ 3x_2 + 3x_3 &= 0 \\ x_3 + 4x_4 &= -1 \\ -3x_1 + 2x_2 + 3x_3 + x_4 &= 5 \end{aligned}$$

$$\begin{bmatrix} 0 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} -3 & 2 & 3 & 1 & 5 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & -4 & -10 \end{bmatrix},$$

$$x_4 = \frac{-10}{-4} = \frac{5}{2}, x_3 = \frac{-1 - 4 \cdot \frac{5}{2}}{1} = -11, x_2 = \frac{0 - 3 \cdot (-11)}{3} = 11, x_1 = \frac{5 - 2 \cdot 11 - 3 \cdot (-11) - 1 \cdot \frac{5}{2}}{-3} = -\frac{9}{2}$$

7. Key statements from Lay, Section 1.1 are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.)

a. Every elementary row operation is reversible.

True: "It is important to note that row operations are reversible."

b. A 5×6 matrix has six rows.

False: "If m and n are positive integers, an m x n matrix is a rectangular array of numbers with m rows and n columns."

c. The solution set of a linear system involving variables $x_1,\ldots,x_n$ is a list of numbers $(s_1,\ldots,s_n)$ that makes each equation in the system a true statement when the values $s_1,\ldots,s_n$ are substituted for $x_1,\ldots,x_n$, respectively.

> True: "A solution of the system is a list $(s_1,\ldots,s_n)$ of numbers that makes each equation a true statement when the values $s_1,\ldots,s_n$ are substituted for $x_1,\ldots,x_n$, respectively."

d. Two fundamental questions about a linear system involve existence and uniqueness.

> True: "TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM
> 1. Is the system consistent; that is, does at least one solution exist?
> 2. If a solution exists, is it the only one; that is, is the solution unique?"

e. Two matrices are row equivalent if they have the same number of rows.

> False: $x_1 = 1$ and $x_1 = 0$ have the same number of rows but are not row equivalent since no elementary row operation converts the first to the second.

f. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

> True: "Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system.

g. Two equivalent linear systems can have different solution sets.

> False: "Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system.

h. A consistent system of linear equations has one or more solutions.

> True: "A system of linear equations is said to be consistent if it has either one solution or infinitely many solutions;..."

Row reduce the matrices in Problems 8 and 9 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

8.
$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} (1) & 2 & 0 & -8 \\ 0 & 0 & (1) & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1 and 3.

9.
$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -18 \\ 0 & 0 & -3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} (1) & 0 & 0 & 1 \\ 0 & (1) & 0 & -2 \\ 0 & 0 & (1) & 2 \end{bmatrix}$$

The pivot columns are 1, 2, and 3.

Find the general solutions of the systems whose augmented matrices are given in Problems 10 - 13.

10.
$$\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & -2 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

x_3 is free, $x_2 = 3$, $x_1 = 4$.

11.

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & -7 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$x_3 = -2$, x_2 is free. $x_1 = 2 - (-2)x_2 = 2 + 2x_2$.

12.

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2/3 & 4/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, x_2 is free. $x_1 = 0 - (-2/3)x_2 - 4/3 \cdot x_3 = 2/3x_2 - 4/3x_3$.

13.

$$\begin{bmatrix} 1 & 0 & 5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_5 = 0$, x_4 is free, x_3 is free. $x_2 = 6 - 4 \cdot x_3 - (-1) \cdot x_4 = 6 - 4x_3 + x_4, x_1 = 3 - 5 \cdot x_3$.

14. Key statements from Lay, Section 1.2 are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, cite an example that shows the statement is not true in all cases or give the location of a statement that has been quoted or used incorrectly.)

a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.

> **False.** "Each matrix is row equivalent to one and only one reduced echelon matrix."

b. The row reduction algorithm applies only to augmented matrices for a linear system.

> **False.** "The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system."

c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

> **True.** "The variables ... corresponding to pivot columns in the matrix are called basic variables."

d. Finding a parametric description of the solution set of a linear system is the same as solving the system.

> **True.** "Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty."

e. If one row in an echelon form of an augmented matrix is [0 0 0 5 0] , then the associated linear system is inconsistent.

> **False.** "A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad \cdots \quad 0 \quad b]$ with b nonzero."