1. What are these sets? Write them using braces, commas, numerals, … (for infinite sets), and $\varnothing$ only.

   (a) $(\{1, 3, 5\} \cup \{3, 1\}) \cap \{3, 5, 7\}$

   (b) $\cup\{\{3\}, \{3, 5\}, \cap\{\{5, 7\}, \{7, 9\}\}\}$

   (c) $(\{1, 2, 5\} - \{5, 7, 9\}) \cup (\{5, 7, 9\} - \{1, 2, 5\})$

   (d) $2^{\{7, 8, 9\}} - 2^{\{7, 9\}}$

   (e) $2^{\varnothing}$

   (f) $\{x : \exists y \in N \text{ where } x = y^2\}$

   (g) $\{x : x \text{ is an integer and } x^2 = 2\}$

2. Prove each of the following:

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (c) $A \cap (A \cup B) = A$

   (d) $A \cup (A \cap B) = A$

   (e) $A - (B \cap C) = (A - B) \cup (A - C)$

3. Write each of the following explicitly:

   (a) $\{1\} \times \{1, 2\} \times \{1, 2, 3\}$

   (b) $\varnothing \times \{1, 2\}$

   (c) $2^{\{1,2\}} \times \{1, 2\}$

4. Let R = {(a, b), (a, c), (c, d), (a, a), (b, a)}. What is R $\circ$ R, the composition of R with itself? What is R^{-1}, the inverse of R? Is R, R $\circ$ R, or R^{-1} a function?

5. What is the cardinality of each of the following sets? Justify your answer.

   (a) S = N - {2, 3, 4}

   (b) S = $\{\varnothing, \{\varnothing\}\}$

   (c) S = $2^{\{a, b, c\}}$

   (d) S = {a, b, c} $\times$ {1, 2, 3, 4}

   (e) S = {a, b, … , z} $\times$ N

6. Consider the chart of example relations in Section 3.2. For the first six, give an example that proves that the relation is missing each of the properties that the chart claims it is missing. For example, show that Mother-of is not reflexive, symmetric, or transitive.

7. Let A, B be two sets. If $2^A = 2^B$, must A = B? Prove your answer.

8. For each of the following sets, state whether or not it is a partition of {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

   (a) {{0}, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}

   (b) {$\varnothing$, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}

   (c) {{1, 2}, {3, 4}, {5, 6}, {7, 8}, {9, 10}}

   (d) {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 6}, {6, 7}, {7, 8}, {8, 9}, {9, 10}}

9. For each of the following relations, state whether it is a partial order (that is not also total), a total order, or neither. Justify your answer.

   (a) DivisibleBy, defined on the natural numbers. (x, y) $\in$ DivisibleBy iff x is evenly divisible by y. So, for example, (9, 3) $\in$ DivisibleBy but (9, 4) $\notin$ DivisibleBy.

(b) LessThanOrEqual defined on ordered pairs of natural numbers. $(a, b) \leq (x, y)$ iff $a \leq x$ or ($a = x$ and $b \leq y$). For example, $(1,2) \leq (2,1)$ and $(1,2) \leq (1,3)$.

(c) The relation defined by the following boolean matrix:

1		1	
	1	1	
		1	1
1			1

10. Are the following sets closed under the following operations? If not, what are the respective closures?
   (a) The odd integers under multiplication.
   (b) The positive integers under division.
   (c) The negative integers under subtraction.
   (d) The negative integers under multiplication.
   (e) The odd length strings under concatenation.

11. What is the reflexive transitive closure R* of the relation
   R = {(a, b), (a, c), (a, d), (d, c), (d, e)} Draw a directed graph representing R*.

12. For each of the following relations R, over some domain D, compute the reflexive, symmetric, transitive closure R′. Try to think of a simple descriptive name for the new relation R′. Since R′ must be an equivalence relation, describe the partition that R induces on D.
   (a) Let D be the set of 50 states in the US. $\forall xy$, xRy iff x shares a boundary with y.
   (b) Let D be the natural numbers. $\forall xy$, xRy iff $y = x+3$.
   (c) Let D be the set of strings containing no symbol except a. $\forall xy$, xRy iff $y = xa$. (i.e., if y equals x concatenated with a).

13. Consider an infinite rectangular grid (like an infinite sheet of graph paper). Let S be the set of intersection points on the grid. Let each point in S be represented as a pair of (x,y) coordinates where adjacent points differ in one coordinate by exactly 1 and coordinates increase (as is standard) as you move up and to the right.
   (a) Let R be the following relation on S: $\forall(x_1,y_1)(x_2,y_2)$, $(x_1,y_1)R(x_2,y_2)$ iff $x_2 = x_1+1$ and $y_2=y_1+1$. Let R′ be the reflexive, symmetric, transitive closure of R. Describe in English the partition P that R′ induces on S. What is the cardinality of P?
   (b) Let R be the following relation on S: $\forall(x_1,y_1)(x_2,y_2)$, $(x_1,y_1)R(x_2,y_2)$ iff $(x_2 = x_1+1$ and $y_2=y_1+1)$ or $(x_2= x_1-1$ and $y_2=y_1+1)$. Let R′ be the reflexive, symmetric, transitive closure of R. Describe in English the partition P that R′ induces on S. What is the cardinality of P?
   (c) Let R be the following relation on S: $\forall(x_1,y_1)(x_2,y_2)$, $(x_1,y_1)R(x_2,y_2)$ iff (x_2,y_2) is reachable from (x_1,y_1) by moving two squares in any one of the four directions and then one square in a perpendicular direction. Let R′ be the reflexive, symmetric, transitive closure of R. Describe in English the partition P that R′ induces on S. What is the cardinality of P?

14. Is the transitive closure of the symmetric closure of a binary relation necessarily reflexive? Prove it or give a counterexample.

15. Give an example of a binary relation that is not reflexive but has a transitive closure that is reflexive.

16. For each of the following functions, state whether or not it is (i) one-to-one, (ii) onto, and (iii) idempotent. Justify your answers.
   (a) $+: P \times P \rightarrow P$, where P is the set of positive integers, and
      $+(a, b) = a + b$ (In other words, simply addition defined on the positive integers)
   (b) $X : B \times B \rightarrow B$, where B is the set {True, False}

X(a, b) = the exclusive or of a and b

17. Consider the following set manipulation problems:
  (a) Let S = {a, b}. Let T = {b, c}. List the elements of P, defined as
  $$P = 2^S \cap 2^T.$$
  (b) Let Z be the set of integers. Let S = {x ∈ Z: ∃y ∈ Z and x = 2y}. Let T = {x ∈ Z: ∃y ∈ Z and x = 2^y}.
Let W = S – T. Describe W in English. List any five consecutive elements of W. Let X = T – S. What is X?

Solutions

1. (a) {3, 5}
  (b) {3, 5, 7}
  (c) {1, 2, 7, 9}
  (d) {8}, {7, 8}, {8, 9}, {7, 8, 9}
  (e) {∅}
  (f) {0, 1, 4, 9, 25, 36…} (the perfect squares)
  (g) ∅ (since the square root of 2 is not an integer)

2. (a) $A \cup (B \cap C)$ $= (B \cap C) \cup A$ commutativity
                                $= (B \cup A) \cap (C \cup A)$ distributivity
                                $= (A \cup B) \cap (A \cup C)$ commutativity

  (b) $A \cap (B \cup C)$ $= (B \cup C) \cap A$ commutativity
                                $= (B \cap A) \cup (C \cap A)$ distributivity
                                $= (A \cap B) \cup (A \cap C)$ commutativity

  (c) $A \cap (A \cup B)$ $= (A \cup B) \cap A$ commutativity
                                $= A$ absorption

3. (a) {(1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,2),, (1,2,3)}
  (b) ∅
  (c) {(∅,1), (∅,2), ({1}, 1), ({1}, 2), ({2}, 1), ({2}, 2), ({1,2}, 1), ({1,2}, 2)}

4.   R ° R = {(a, a), (a, d), (a, b), (b, b), (b, c), (b, a), (a, c)}
        R inverse = {(b, a), (c, a), (d, c), (a, a), (a, b)}
        None of R, R ° R or R inverse is a function.

5. (a) S = {0, 1, 5, 6, 7, …}. S has the same number of elements as N. Why? Because there is a bijection
between S and N: f: S → N, where f(0) = 0, f(1) = 1, ∀x ≥ 5, f(x) = x - 3. So |S| = $\aleph_0$.
  (b) 2.
  (c) S = all subsets of {a, b, c}. So S = {∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}. So |S| = 8. We
       could also simply have used the fact that the cardinality of the power set of a finite set of cardinality c
       is 2^c.
  (d) S = {(a, 1), (a, 2), (a, 3), (a, 4), (b, 1), (b, 2), (b, 3), (b, 4), (c, 1), (c, 2), (c, 3), (c, 4)}. So |S| = 12. Or
       we could have used the fact that, for finite sets, $|A \times B| = |A| * |B|$.
  (e) S = {(a, 0), (a, 1), …, (b, 0), (b, 1),…} Clearly S contains an infinite number of elements. But are there
       the same number of elements in S as in N, or are there more (26 times more, to be precise)? The
       answer is that there are the same number. |S| = $\aleph_0$. To prove this, we need a bijection from S to N. We
       can define this bijection as an enumeration of the elements of S:
           (a, 0), (b, 0), (c, 0), …   (First enumerate all 26 elements of S that have 0 as their second element)

(a, 1), (b, 1), (c, 1), …   (Next enumerate all 26 elements of S that have 1 as their second element)
and so forth.

6. Mother-of: Not reflexive: Eve is not the mother of Eve (in fact, no one is her own mother).
                    Not symmetric: mother-of(Eve, Cain), but not Mother-of(Cain, Eve).
                    Not transitive: Each person has only one mother, so if Mother-of(x, y) and Mother-of(y, z),
                       the only way to have Mother-of(x, z) would be if x and y are the same person, but we know
                       that that's not possible since Mother-of(x, y) and no one can be the mother of herself).
   Would-recognize-picture-of:
                    Not symmetric: W-r-p-o(Elaine, Bill Clinton), but not W-r-p-o (Bill Clinton, Elaine)
                    Not transitive: W r-p-o(Elaine, Bill Clinton) and W r-p-o(Bill Clinton, Bill's mom) but not
                                    W-r-p-o(Elaine, Bill's mom)
   Has-ever-been-married-to: Not reflexive: No one is married to him or herself.
                                   Not transitive: H-e-b-m-t(Dave, Sue) and H-e-b-m-t(Sue, Jeff) but not
                                                   H-e-b-m-t(Dave, Jeff)
   Ancestor-of: Not reflexive: not Ancestor-of(Eve, Eve) (in fact, no one is their own ancestor).
                  Not symmetric: Ancestor-of(Eve, Cain) but not Ancestor-of(Cain, Eve)
   Hangs-out-with: Not transitive: Hangs-out-with(Bill, Monica) and Hangs-out-with(Monica, Linda Tripp),
                     but not Hangs-out-with(Bill, Linda Tripp).
   Less-than-or-equal-to: Not symmetric: $1 \leq 2$, but not $2 \leq 1$.

7. Yes, if $2^A = 2^B$, then A must equal B. Suppose it didn't. Then there is some element x that is in one set but not the other. Call the set x is in A. Then 2^A must contain $\{x\}$, which must not be in 2^B, since $x \notin B$. This would mean that $2^A \neq 2^B$, which contradicts our premise.

8. (a)   yes
   (b)   no, since no element of a partition can be empty.
   (c)   no, 0 is missing
   (d)   no, since, each element of the original set S must appear in only one element of a partition of S.

9. (a)   DivisibleBy is a partial order. $\forall x \ (x, x) \in$ DivisibleBy, so DivisibleBy is reflexive. For x to be DivisibleBy y, x must be greater than or equal to y. So the only way for both (x, y) and (y, x) to be in DivisibleBy is for x and y to be equal. Thus DivisibleBy is antisymmetric. And if x is DivisibleBy y and y is DivisibleBy z, then x is DivisibleBy z. So DivisibleBy is transitive. But DivisibleBy is not a total order. For example neither (2, 3) nor (3, 2) is in it.
   (b)   LessThanOrEqual defined on ordered pairs is a total order. This is easy to show by relying on the fact that $\leq$ for the natural numbers is a total order.
   (c)   This one is not a partial order at all because, although it is reflexive and antisymmetric, it is not transitive. For example, it includes (4, 1) and (1, 3), but not (4, 3).

10. (a) The odd integers are closed under multiplication. Every odd integer can be expressed as 2n+1 for some value of $n \in N$. So the product of any two odd integers can be written as (2n+1)(2m+1) for some values of n and m. Multiplying this out, we get 4(n+m) +2n + 2m +1, which we can rewrite as 2(2(n+m) + n + m) +1, which must be odd.
   (b) The positive integers are not closed under division. To show that a set is not closed under an operation, it is sufficient to give one counterexample. 1/2 is not an integer. The closure of the positive integers under division is the positive rationals.
   (c) The negative integers are not closed under subtraction. -2 - (-4) = 2. The closure of the negative numbers under subtraction is the integers.
   (d) The negative integers are not closed under multiplication. -2 * -2 = 4. The closure of the negative numbers under multiplication is the nonzero integers. Remember that the closure is the *smallest* set that

contains all the necessary elements. Since it is not possible to derive zero by multiplying two negative numbers, it must not be in the closure set.

(e) The odd length strings are not closed under concatenation. "a" || "b" = "ab", which is of length 2. The closure is the set of strings of length ≥ 2. Note that strings of length 1 are not included. Why?

11. R* = R ∪ {(x, x) : x ∈ {a, b, c, d, e}} ∪ {(a, e)}

12. (a) The easiest way to start to solve a problem like this is to start writing down the elements of R′ and see if a pattern emerges. So we start with the elements of R: {(TX, LA), (LA, TX), (TX, NM), (NM, TX), (LA, Ark), (Ark, LA), (LA Miss), (Miss, LA) …}. To construct R′, we first add all elements of the form (x, x), so we add (TX,TX), and so forth. Then we add the elements required to establish transitivity:

(NM, TX), (TX, LA) ⇒ (NM, TX)

(TX, LA), (LA, Ark) ⇒ (TX, Ark)

(NM, TX), (TX, Ark) ⇒ (NM, Ark), and so forth.

If we continue this process, we will see that the reflexive, symmetric, transitive closure R′ relates all states except Alaska and Hawaii to each other and each of them only to themselves. So R′ can be described as relating two states if it's possible to drive from one to the other without leaving the country. The partition is:

[Alaska]

[Hawaii]

[all other 48 states]

(b) R includes, for example {(0, 3), (3, 6), (6, 9), (9, 12) …}. When we compute the transitive closure, we add, among other things {(0, 6), (0, 9), (0,12)}. Now try this starting with (1,4) and (2, 5). It's clear that ∀x,y, xR′y iff x = y (mod 3). In other words, two numbers are related iff they have the same remainder mod 3. The partition is:

[0, 3, 6, 9, 12 …]

[1, 4, 7, 10, 13 …]

[2, 5, 8, 11, 14 …]

(c) R′ relates all strings composed solely of a's to each other. So the partition is

[ε, a, aa, aaa, aaaa, …]

13. (a) Think of two points being related via R if you can get to the second one by starting at the first and moving up one square and right one square. When we add transitivity, we gain the ability to move diagonally by two squares, or three, or whatever. So P is an infinite set. Each element of P consists of the set of points that fall on an infinite diagonal line running from lower left to upper right.

(b) Now we can more upward on either diagonal. And we can move up and right followed by up and left, and so forth. The one thing we can't do is move directly up or down or right or left exactly one square. So take any given point. To visualize the points to which it is related under R′, imagine a black and white chess board where the squares correspond to points on our grid. Each point is related to all other points of the same color. Thus the cardinality of P is 2.

(c) Now every point is related to every other point. The cardinality of P is 1.

14. You might think that for all relations R on some domain D, the transitive closure of the symmetric closure of R (call it TC(SC(R))) must be reflexive because for any two elements x, y ∈ D such that (x, y) ∈ R, we'll have (x, y), (y, z) ∈ SC(R) and therefore (x, x), (y, y) ∈ TC(SC(R)). This is all true, but does not prove that for all z ∈ D, (z, z) ∈ TC(SC(R)). Why not? Suppose there is a z ∈ D such that there is no y ∈ D for which (y, z) ∈ R or (z, y) ∈ R. (If you look at the graph of R, z is an isolated vertex with no edges in or out.) Then (z, z) ∉ TC(SC(R)). So the answer is no, with R = ∅ on domain {a} as a simple counterexample: TC(SC(R)) = ∅, yet it should contain (a, a) if it were reflexive.

15. R = {(a, b), (b, a)} on domain {a, b} does the trick easily.

16. (a) (i) + is not one-to-one. For example +(1, 3) = +(2, 2) = 4.

(ii) + is not onto. There are no two positive integers that sum to 1.

(iii) + is not idempotent. +(1, 1) ≠ 1.

(b) (i) X is not one-to-one. For example True X True = False X False = False.

(ii) X is onto. Proof: True X True = False. True X False = True. In general, when the domain is a finite set, it's easy to show that a function is onto: just show one way to derive each element.

(iii) X is not idempotent. True X True = False.

17. (a) P = {∅,{b}}

(b) S is the set of even numbers. T is the set powers of 2. W is the set of even numbers that are not powers of 2. So W = { ….-6, -4, -2, 0, 6, 10, 12, 14, 18, …}. X is the set of numbers that are powers of 2 but are not even. There's only one element of X. X = {1}.