M 340L – CS
Homework Set 9 Solutions

1. [15] a. A projection matrix is a matrix P such that $P^2 = P$. Using that, prove that if a (possibly rectangular) matrix Q satisfies $Q^TQ = I$, then QQ^T is a projection matrix.

Since $(QQ^T)^2 = QQ^TQQ^T = Q(Q^TQ)Q^T = QQ^T$, QQ^T is a projection.

b. Using the definition of an inverse, prove for any invertible matrices A and B, that $(AB)^{-1} = B^{-1}A^{-1}$.

Since $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$, $B^{-1}A^{-1}$ must be the inverse of AB.

2. [15] Given $A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ -4 & 4 \end{bmatrix}$, use the Gram Schmidt algorithm to express $A = QR$, where $Q^TQ = I$ and R is upper triangular.

$$R_{1,1} = \left\| \begin{matrix} 2 \\ 4 \\ -4 \end{matrix} \right\| = 6,\ Q_{.,1} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, R_{1,2} = Q_{.,1}^T A_{.,2} = -3,\ \bar{Q}_{.,2} = A_{.,2} - R_{1,2}Q_{.,1} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix},$$

$$R_{2,2} = \|\bar{Q}_{.,2}\| = 2\sqrt{2}, Q_{.,2} = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}. \text{Thus, } A = \begin{bmatrix} 1/3 & 0 \\ 2/3 & \sqrt{2}/2 \\ -2/3 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 0 & 2\sqrt{2} \end{bmatrix}.$$

3. [20] Identify which of the following satisfy the definition for vector spaces. For each case either mark either "Yes" or "No" in the columns "Closed under addition" and "Closed under scalar multiplication". For each answer of "No", give a simple example showing a failure of the property.

	Closed under addition	Closed under scalar multiplication
a. The set of two by two matrices A such that $A_{1,1}A_{2,2} \geq 0$.	__no____	___yes____

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \text{are in the set but not } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b. The set of functions $f : \mathbb{R} \to [0,1]$ (Note: this is **not** $f : [0,1] \to \mathbb{R}$)	___no____	___no____

f such that for all $x, f(x) = 1$ is in the set but not $f + f \equiv 2$.
f such that for all $x, f(x) = 1$ is in the set but not $-f \equiv -1$.

c. The set of 4-vectors lying on a plane in $\mathbb{R}^4$ that includes the origin.	___yes____	___yes___
d. The set of ordered triples of real numbers (a,b,c) so that $a \geq 0, b \geq 0,$ and $c \geq 0$.	___yes____	___no____

$(1,0,0)$ is in the set but not $-1 \cdot (0,1,0) = (-1,0,0)$.

e. The null space of a 4 by 2 matrix A.	___yes____	___yes____

4. [10] Answer true or false:

true **a.** Given nonzero vectors u and v, the matrix uv^T has a null space equal to all vectors perpendicular to v.

_true _ **b.** If both $Ax = b$ and $A\overline{x} = b$ then $x - \overline{x}$ must be in the null space of A.

_true _ **c.** If the columns of the square matrix A form a basis for $\mathbb{R}^m$ then A is invertible.

_true _ **d.** For any square matrix A, if for some nonzero vector x, $Ax = 0$ then the **rows** of A are linearly dependent.

5. [10] Prove that if $A = QR$, where Q is square, $Q^T Q = I$, and R is m by n upper triangular then $\|Ax - b\| = \|Rx - Q^T b\|$.

Since Q is square and $Q^T Q = I$, $Q^T = Q^{-1}$ and $QQ^T = I$. Thus, Q^T is also orthogonal and $\|Ax - b\| = \|Q^T(Ax - b)\|$ since norms are preserved through orthogonal transformation. But then $\|Ax - b\| = \|Q^T(Ax - b)\| = \|Q^T(QRx - b)\| = \|Q^T QRx - Q^T b\| = \|Rx - Q^T b\|$.

6. [15] Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$ or show that no inverse exists.

We need to solve the systems $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for the columns of the inverse matrix $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$. We transform

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{bmatrix}$$ and see

that the system matrix has linearly dependent columns and thus no inverse exists

7. **[15]** Suppose that applying the Gram-Schmidt algorithm to $A=\begin{bmatrix}-2 & 4\\ 1 & 4\\ 0 & 2\\ 2 & -7\end{bmatrix}$ results in

$A=QR,$ where $Q=\begin{bmatrix}-2/3 & 0\\ 1/3 & 6/7\\ 0 & 2/7\\ 2/3 & -3/7\end{bmatrix}$ and $R=\begin{bmatrix}3 & -6\\ 0 & 7\end{bmatrix}$. (The matrix Q satisfies $Q^TQ=I$.)

Determine x^* that minimizes $\|Ax-b\|$ over all $x\in\mathbb{R}^2$, where $b=\begin{bmatrix}0\\ 21\\ 0\\ -42\end{bmatrix}$.

Hint: A smart student would now test A^Tr, where $r=Ax^*-b$.

First we compute $c=Q^Tb=Q=\begin{bmatrix}-2/3 & 0\\ 1/3 & 6/7\\ 0 & 2/7\\ 2/3 & -3/7\end{bmatrix}^T\begin{bmatrix}0\\ 21\\ 0\\ -42\end{bmatrix}=\begin{bmatrix}-21\\ 36\end{bmatrix}$. Then we solve

$Rx^*=c$ to get $x_2^*=36/7$ and $x_1^*=(-21-(-6)36/7)/3=23/7$. So $x^*=\begin{bmatrix}23/7\\ 36/7\end{bmatrix}$.

If we test A^Tr we get

$$r=Ax^*-b=\begin{bmatrix}-2 & 4\\ 1 & 4\\ 0 & 2\\ 2 & -7\end{bmatrix}\begin{bmatrix}23/7\\ 36/7\end{bmatrix}-\begin{bmatrix}0\\ 21\\ 0\\ -42\end{bmatrix}=\begin{bmatrix}14\\ 167/7\\ 72/7\\ -206/7\end{bmatrix}-\begin{bmatrix}0\\ 21\\ 0\\ -42\end{bmatrix}=\begin{bmatrix}14\\ 20/7\\ 72/7\\ 88/7\end{bmatrix}$$ and

$$A^Tr=\begin{bmatrix}-2 & 4\\ 1 & 4\\ 0 & 2\\ 2 & -7\end{bmatrix}^T\begin{bmatrix}14\\ 20/7\\ 72/7\\ 88/7\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$