M 340L – CS
Homework Set 11

Note: Scale all eigenvectors so the largest component is + 1.

1. How do perturbations affect eigenvalues and eigenvectors?

a. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (See the note above regarding the scaling of eigenvectors and make sure you do it throughout the homework.)

b. Let $B = \begin{bmatrix} 2 & 1 \\ 0 & 2+\varepsilon \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Your answers should be in terms of the perturbation parameter ε.)

c. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of A. Comment on the linear independence of the eigenvectors of B.

d. Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors? (Choose linearly independent eigenvectors.)

e. Let $D = \begin{bmatrix} 2 & \varepsilon \\ 0 & 2 \end{bmatrix}$. What are its eigenvalues and eigenvectors?

f. Describe the effect of the perturbation ε on eigenvalues and eigenvectors of C.

2. Using the diagonal form to compute high powers:

Let $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$. *Feel free to express answers in parts **c**, **d**, and **e** using algebraic expressions involving powers.*

a. What are its eigenvalues and eigenvectors?

b. Using part **a.** form D, a diagonal matrix of eigenvalues, form V whose columns are the associated eigenvectors, then compute V^{-1}, and finally VDV^{-1}. Compare VDV^{-1} to A.

c. Using part **b**. and the fact that $A^k = VD^kV^{-1}$, what is $A^{100}y$, for $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$? (Do **not** compute A^{100} - yet. Use associativity in a clever way.)

d. Express your answer in part **c** as $A^{100}y = \gamma\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is + 1. Compare $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to the eigenvector corresponding to λ_1.

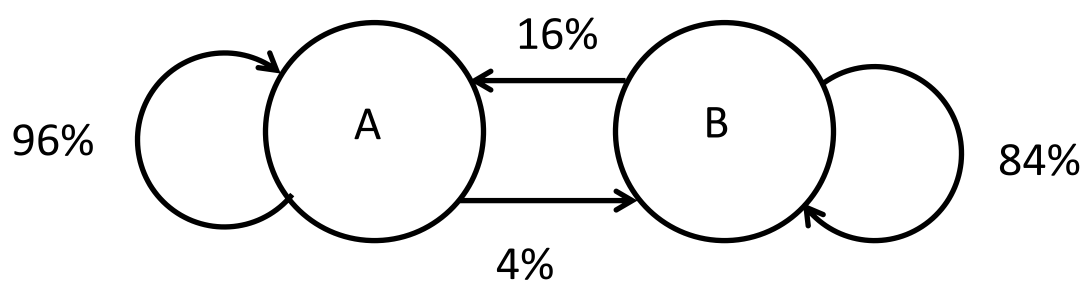
e. Using part **b.**, what is A^{100}?

3. All zero eigenvalues:

Find a simple **non-zero** matrix having all zero eigenvalues.

4. A Markov process:

A city has two restaurants: A and B. 96% of the time, a person leaving A will return to A the next time she goes to either A or B. Thus, 4% of the time, she will switch to B the next time. 84% of the time, a person leaving B will return to B the next time she goes to either A or B. Thus, 16% of the time, she will switch to A the next time. This diagram summarizes the situation:



Let $A = \begin{bmatrix} 24/25 & 4/25 \\ 1/25 & 21/25 \end{bmatrix}$ (i.e. the matrix of transitions). use $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

a. What are its eigenvalues and eigenvectors?

b. Using part a. express $A = VDV^{-1}$ (where the columns of V are the eigenvectors and D is a diagonal matrix containing the associated eigenvalues.)

c. Using the fact that $A^k = VD^kV^{-1}$, what is $A^{100}y$, for $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$? (See comments in 1c.)

d. Express your answer in part **c** as $A^{100}y = \gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where γ is such that the largest component of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is + 1. Compare $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ to the eigenvector corresponding to λ_1.

e. Using part **b.**, what is A^{100}?