3-message protocols that satisfy completeness, special soundness, and HVZK are called Σ-protocols

↳ Σ-protocols are useful for building signatures and identification protocols

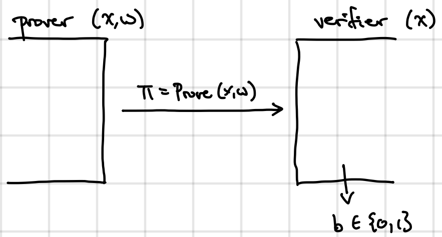
How can a prover both <u>prove knowledge</u> and yet be zero-knowledge at the same time?

↳ Extractor operates by "rewinding" the prover (if the prover has good success probability, it can answer most challenges correctly.

↳ But in the real (actual) protocol, verifier <u>cannot</u> rewind (i.e., verifier only sees prover on fresh protocol executions), which can provide zero-knowledge.

Many extensions of Schnorr's protocol to prove relations in the exponent.

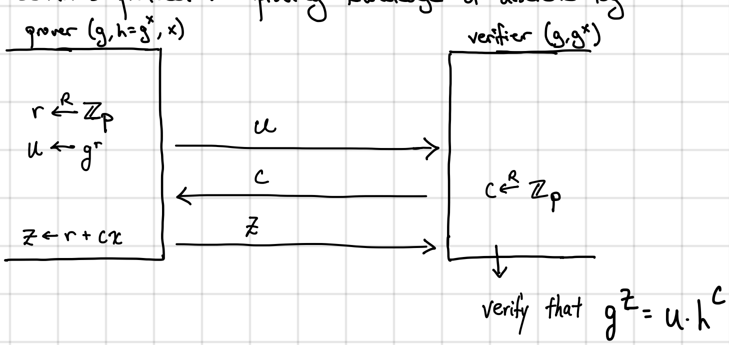
<u>Non-interactive zero-knowledge</u> (NIZK): Can we construct a zero-knowledge proof system where the proof is a <u>single</u> message from the prover to the verifier?



NIZKs for NP unlikely to exist for NP (unless NP $\subseteq$ BPP), but possible in the random oracle model (as well as in the common reference string model)

<u>Fiat-Shamir heuristic</u>: NIZKs in random oracle model

Recall Schnorr's protocol for proving knowledge of discrete log:



In this protocol, verifier's message is uniformly random (and in fact, is "public coin" — the verifier has <u>no</u> secrets)

<u>Key idea</u>: Replace the verifier's challenge with a hash function $H : \{0,1\}^* \to \mathbb{Z}_p$

Namely, instead of sampling $c \xleftarrow{R} \mathbb{Z}_p$, we sample $c \leftarrow H(g, h, u)$. ← prover can now compute this quantity on its own!

Completess, zero-knowledge, proof of knowledge follow by a similar analysis as Schnorr [will rely on random oracle]

Signatures from discrete log in RO model (Schnorr):

- Setup: $x \xleftarrow{R} \mathbb{Z}_p$

  vk: $(g, h = g^x)$ sk: x

- Sign(sk, m): $r \xleftarrow{R} \mathbb{Z}_p$

  $u \leftarrow g^r$ $c \leftarrow H(g, h, u, m)$ $z \leftarrow r + cx$

  $\sigma = (u, z)$

  } signature is a NIZK proof of knowledge of discrete log of h (with challenge derived from the message m)

- Verify(vk, m, σ): write $\sigma = (u, z)$, vk $= h$, compute $c \leftarrow H(g, h, u, m)$ and accept if $g^z = u \cdot h^c$

Security essentially follows from security of Schnorr's identification protocol (together with Fiat-Shamir)
↳ forged signature on a new message m is a proof of knowledge of the discrete log (can be extracted from adversary)

Length of Schnorr's signature: $vk: (g, h = g^x)$ $\quad \sigma: (g^r, \underbrace{c = H(g, h, g^r, m)}, \; z = r + cx)$ $\quad$ Verification checks that $g^z = g^r h^c$
$sk: x$

can be computed given other components, so do not need to include $\implies |\sigma| = 2 \cdot |\mathbb{G}|$ $\quad$ [512 bits if $|\mathbb{G}| = 2^{256}$]

But, can do better... observe that challenge c only needs to be 128-bits (the knowledge error of Schnorr is $1/|C|$ where C is the set of possible challenges), so we can sample a 128-bit challenge rather than 256-bit challenge. Thus, instead of sending (g^r, z), instead send (c, z) and compute $g^r = g^z / h^c$ and that $c = H(g, h, g^r, m)$. Then resulting signatures are 384 bits

128 bit challenge
+
256 bit group element

Important note: Schnorr signatures are randomized, and security relies on having good randomness
↳ What happens if randomness is reused for two different signatures?

Then, we have

$$\left.\begin{aligned} \sigma_1 &= (g^r, \; c_1 = H(g, h, g^r, m_1), \; z_1 = r + c_1 x) \\ \sigma_2 &= (g^r, \; c_2 = H(g, h, g^r, m_2), \; z_2 = r + c_2 x) \end{aligned}\right\} \; z_1 - z_2 = (c_1 - c_2)x \implies x = (c_1 - c_2)^{-1}(z_1 - z_2)$$

This is precisely the set of relations the knowledge extractor uses to recover the discrete log x (i.e., the signing key)!

Deterministic Schnorr: We want to replace the random value $r \xleftarrow{R} \mathbb{Z}_p$ with one that is deterministic, but which does not compromise security
↳ Derive randomness from message using a PRF. In particular, signing key includes a secret PRF key k, and signing algorithm computes $r \leftarrow F(k, m)$ and $\sigma \leftarrow \mathrm{Sign}(sk, m; r)$.
↳ Avoids randomness reuse/misuse vulnerabilities.

TLS protocol

digital signature algorithm / elliptic-curve DSA

In practice, we use a variant of Schnorr's signature scheme called DSA / ECDSA
↳ larger signatures (2 group elements – 512 bits) and proof only in "generic group" model $\quad$ [but we use it because Schnorr was patented ... until 2008]

ECDSA signatures (over a group $\mathbb{G}$ of prime order p):

- Setup: $x \xleftarrow{R} \mathbb{Z}_p$
  $vk: (g, h = g^x)$ $\quad sk: x$
- Sign(sk, m): $\alpha \xleftarrow{R} \mathbb{Z}_p$
  $u \leftarrow g^\alpha$ $\quad r \leftarrow f(u) \in \mathbb{Z}_p$ $\quad$ deterministic function specified by ECDSA
  $s \leftarrow (H(m) + r \cdot x)/\alpha \in \mathbb{Z}_p$
  $\sigma = (r, s)$

[Specifically, $f(u)$ parses $u = (\hat{x}, \hat{y}) \in \mathbb{F}_q^2$ where $\mathbb{F}_q$ is the base field over which the elliptic curve is defined, and outputs $\hat{x} \pmod p$, where $\hat{x}$ is viewed as a value in $[0, q)$]

- Verify(vk, m, σ): write $\sigma = (r, s)$, compute $u \leftarrow g^{H(m)/s} h^{r/s}$, accept if $r = f(u)$
  $vk = h$

Correctness: $u = g^{H(m)/s} h^{r/s} = g^{[H(m) + rx]/s} = g^{[H(m)+rx]/[H(m)+rx]\alpha^{-1}} = g^\alpha$ and $r = f(g^\alpha)$

Security analysis non-trivial: requires either strong assumptions or modeling $\mathbb{G}$ as an "ideal" group

Signature size: $\sigma = (r, s) \in \mathbb{Z}_p^2$ — for 128-bit security, $p \sim 2^{256}$ so $|\sigma| = 512$ bits (can use P-256 or Curve25519)