CS 346: Introduction to Cryptography

Number Theory and Algebra Fact Sheet

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Groups

- A group $(\mathbb{G}, \star)$ consists of a group $\mathbb{G}$ together with an operation $\star$ with the following properties:
  - **Closure:** If $g, h \in \mathbb{G}$, then $g \star h \in \mathbb{G}$.
  - **Associativity:** For all $g, h, k \in \mathbb{G}$, $g \star (h \star k) = (g \star h) \star k$.
  - **Identity:** There exists an (unique) element $e \in \mathbb{G}$ such that for all $g \in \mathbb{G}$, $e \star g = g = g \star e$.
  - **Inverse:** For every element $g \in \mathbb{G}$, there exists an (unique) element $h \in \mathbb{G}$ where $g \star h = e = h \star g$.
- A group $(\mathbb{G}, \star)$ is **commutative** (or *abelian*) if for all $g, h \in \mathbb{G}$, $g \star h = h \star g$.
- **Notation:** Unless otherwise noted, we will denote the group operation by '$\cdot$' (i.e., multiplicative notation). If $g, h \in \mathbb{G}$, we write gh to denote $g \cdot h$. For a group element $g \in \mathbb{G}$, we write g^{-1} to denote the inverse of g. We write g^0 and 1 to denote the identity element. For a positive integer k, we write g^k to denote
$$g^k := \underbrace{g \cdot g \cdots g}_{k \text{ copies}}.$$
For a negative integer k, we write g^{-k} to denote $(g^k)^{-1}$.
- A group $\mathbb{G}$ is *cyclic* if there exists a *generator* g such that $\mathbb{G} = \left\{g^0, \ldots, g^{|\mathbb{G}|-1}\right\}$.
- For an element $g \in \mathbb{G}$, we write $\langle g \rangle := \left\{g^0, g^1, \ldots, g^{|\mathbb{G}|} - 1\right\}$ to denote the *subgroup generated by* g. The *order* $\mathrm{ord}(g)$ of g in $\mathbb{G}$ is the size of the subgroup generated by g: $\mathrm{ord}(g) := |\langle g \rangle|$. The order of the group $\mathbb{G}$ is the size of the group: $\mathrm{ord}(\mathbb{G}) = |\mathbb{G}|$.
- **Lagrange's theorem:** For a group $\mathbb{G}$ and any element $g \in \mathbb{G}$, the order of g divides the order of the group: $\mathrm{ord}(g) \mid |\mathbb{G}|$.
- If $\mathbb{G}$ is a group of prime order, then $\mathbb{G} = \langle g \rangle$ for every $g \neq 1$ (i.e., every non-identity element of a prime-order group is a generator).

The Groups $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$

- We write $\mathbb{Z}_n$ to denote the group of integers $\mathbb{Z}_n := \{0, 1, \ldots n-1\}$ under addition modulo n.
- We write $\mathbb{Z}_n^*$ to denote the group of integers $\mathbb{Z}_n^* := \{x \in \mathbb{Z}_n : (\exists y \in \mathbb{Z}_n : xy = 1 \bmod n)\}$ under multiplication modulo n.
- **Bezout's identity:** For all integers $x, y \in \mathbb{Z}$, there exists integers $s, t \in \mathbb{Z}$ such that $xs + yt = \gcd(x, y)$.
  - Given x, y, computing s, t can be computed in time $O(\log|x| \cdot \log|y|)$ using the *extended Euclidean algorithm.*

- An element $x \in \mathbb{Z}_n$ is invertible if and only if $\gcd(x, n) = 1$. This gives an equivalent characterization of $\mathbb{Z}_n^*$: $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n : \gcd(x, n) = 1\}$. Computing an inverse of $x \in \mathbb{Z}_n^*$ can be done efficiently via the extended Euclidean algorithm.
- For prime p, the group $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$. The order of $\mathbb{Z}_p^*$ is $\left|\mathbb{Z}_p^*\right| = p - 1$. In particular $\mathbb{Z}_p^*$ is *not* a group of prime order (whenever $p > 3$). Computing the order of an element $g \in \mathbb{Z}_p^*$ is efficient if the factorization of the group order (i.e., $p - 1$)l is known.

- For a positive integer n, *Euler's phi function* (also called *Euler's totient function*) is defined to be the number of integers $1 \leq x \leq n$ where $\gcd(x, n) = 1$. In particular, $\varphi(n)$ is the order of $\mathbb{Z}_n^*$. If $p_1^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}$ is the prime factorization of n, then

$$\varphi(n) = n \cdot \prod_{i \in [\ell]} \left(1 - \frac{1}{p_i}\right) = \prod_{i \in [\ell]} p_i^{k_i - 1}(p_i - 1).$$

- Special cases of Lagrange's theorem:
  - **Fermat's theorem:** For prime p and $x \in \mathbb{Z}_p^*$, $x^{p-1} = 1 \pmod{p}$.
  - **Euler's theorem:** For a positive integer n and $x \in \mathbb{Z}_n^*$, $x^{\varphi(n)} = 1 \pmod{n}$.

Operations over Groups

- Let n be a positive integer. Take any $x, y \in \mathbb{Z}_n$. The following operations can be performed efficiently (i.e., in time $\mathrm{poly}(\log n)$):
  - Sampling a random element $r \xleftarrow{\text{R}} \mathbb{Z}_n$.
  - Basic arithmetic operations: $x + y \pmod{n}, x - y \pmod{n}, xy \pmod{n}, x^{-1} \pmod{n}$. These operations suffice to solve linear systems.
  - Exponentiation: Computing $x^k \pmod{n}$ can be done in $\mathrm{poly}(\log n, \log k)$ time using repeated squaring.

- Suppose $N = pq$ where p, q are two large primes. Let $x \in \mathbb{Z}_n$. Then, the following problems are believed to be hard:
  - Finding the prime factors of N. This is equivalent to the problem of computing $\varphi(N)$.
  - Computing an e^{th} root of x where $\gcd(N, e) = 1$ (i.e., a value y such that $x^e = y \bmod N$).

- Let $\mathbb{G}$ be a group of prime order p with generator g. We often consider the following computational problems over $\mathbb{G}$:
  - **Discrete logarithm:** Given (g, h) where $h = g^x$ and $x \xleftarrow{\text{R}} \mathbb{Z}_p$, compute x.
  - **Computational Diffie-Hellman (CDH):** Given (g, g^x, g^y) where $x, y \xleftarrow{\text{R}} \mathbb{Z}_p$, compute g^{xy}.
  - **Decisional Diffie-Hellman (DDH):** Distinguish between (g, g^x, g^y, g^{xy}) and (g, g^x, g^y, g^r) where $x, y, r \xleftarrow{\text{R}} \mathbb{Z}_p$.