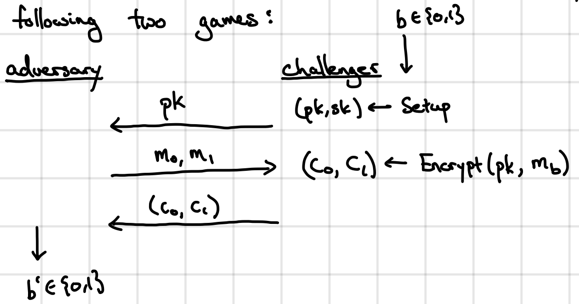
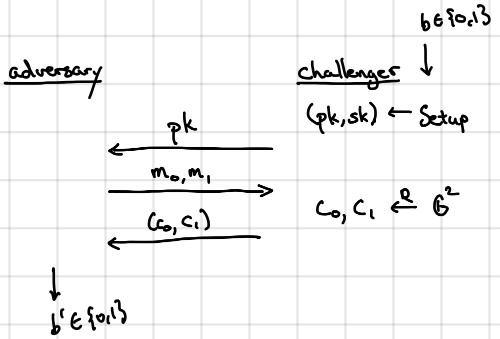
Security: If DDH holds in $\mathbb{G}$, then ElGamal is semantically secure.

Proof. Consider following two games:

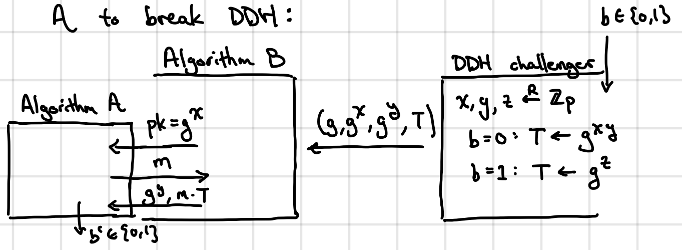

$b \in \{0,1\}$
adversary
challenger
pk
$(pk, sk) \leftarrow \text{Setup}$
m_0, m_1
$(c_0, c_1) \leftarrow \text{Encrypt}(pk, m_b)$
(c_0, c_1)
$b' \in \{0,1\}$



adversary's advantage in guessing b is 0 here since (c_0, c_1) is independent of (m_0, m_1)!

Claim: these two games are indistinguishable under DDH

Proof. Suppose there exists efficient A that can distinguish $(c_0, c_1) \leftarrow \text{Encrypt}(pk, m)$ from $(c_0, c_1) \xleftarrow{R} \mathbb{G}^2$. We use A to break DDH:



Observe: x is uniform over $\mathbb{Z}_p$ so g^x is a properly-generated public key (for ElGamal)

if $T = g^{xy}$, then $(g^y, T \cdot m) = (g^y, g^{xy} \cdot m)$ which is the output of Encrypt(pk, m) with randomness y — this is exactly the distribution where A sees Encrypt(pk, m)

if $T = g^z$, then $(g^y, g^z \cdot m)$ is uniform over $\mathbb{G}^2$ (since y, z are sampled independently of each other and of m) — this is exactly the distribution where A sees $(c_0, c_1) \xleftarrow{R} \mathbb{G}^2$

distinguishing advantage of B = distinguishing advantage of A

Equivalent view: Under DDH, g^{xy} looks uniform even given g, g^x, g^y, so an ElGamal ciphertext looks indistinguishable (to an efficient adversary) from a OTP encryption

What if we want to encrypt longer messages? [or messages that is not a group element]

- Hybrid encryption (key encapsulation [KEM]):

  Use PKE scheme to encrypt a secret key

  Encrypt payload using secret key + authenticated encryption

  called <u>key encapsulation</u>

  PKE. Encrypt(pk, k) "header" [slow]

  AE. Encrypt(k, m) "payload" [fast]

  secret-key operations much much faster than public-key operations!

- How to derive key from group element?

  Same as in key-exchange: hash the group element to a bit-string (symmetric key)

  e.g., Hash-ElGamal: Encrypt(pk, m): $y \xleftarrow{R} \mathbb{Z}_p$

  $c = (g^y, m \oplus H(g, h, g^y, h^y))$

  as before, can also rely on CDH + ideal hash function (random oracle)

  $H : \mathbb{G}^4 \rightarrow \{0,1\}^n$

Vanilla ElGamal described above is <u>not</u> CCA-secure!

Ciphertexts are malleable: given $ct = (g^y, h^y \cdot m)$, can construct ciphertext $(g^y, h^y \cdot m \cdot g)$ which decrypts to message $m \cdot g$

↳ directly implies a CCA attack

Several approaches to get CCA security from DH assumptions:

- Cramer-Shoup (CCA-security from DDH) - based on hash-proof systems
- Fujisaki-Okamoto transformation (using an ideal hash function + CDH)
- Make stronger assumption ("interactive" CDH + use ideal hash function):

We do <u>not</u> know of any groups where CDH believed to be hard, but interactive CDH is easy.

"CDH is hard even given access to a DDH oracle"

also called strong DH assumption

  - Setup: $x \xleftarrow{R} \mathbb{Z}_p$ $\quad$ $pk: h$
    $h \leftarrow g^x$ $\quad$ $sk: x$

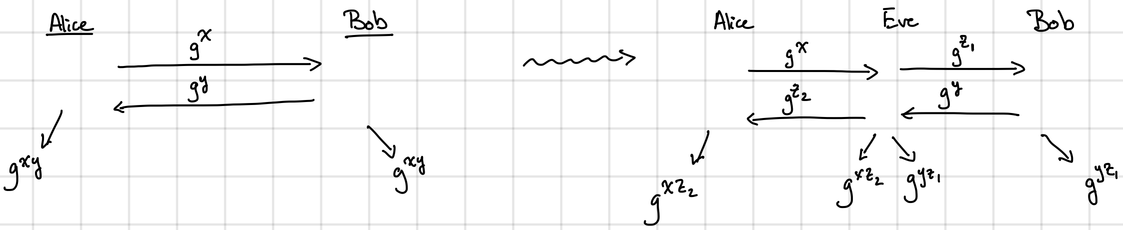
symmetric authenticated encryption scheme

  - Encrypt(pk, m): $y \xleftarrow{R} \mathbb{Z}_p$ $\quad$ $k \leftarrow H(g, g^x, g^y, h^y)$ $\quad$ $ct' \leftarrow \text{Enc}_{AE}(k, m)$
    $c \leftarrow (g^y, ct')$
  - Decrypt(sk, c): $k \leftarrow H(g, g^x, c_0, c_0^x)$
    $m \leftarrow \text{Dec}_{AE}(k, c_1)$

Essentially ElGamal where key derived from hash function

Diffie-Hellman key-exchange is an anonymous key-exchange protocol: neither side knows who they are talking to

↳ vulnerable to a "man-in-the-middle" attack



Observe Eve can now decrypt all of the messages between Alice and Bob and Alice + Bob have no idea!

What we require: authenticated key-exchange (not anonymous) and relies on a root of trust (e.g., a certificate authority)

↳ On the web, one of the parties will authenticate themself by presenting a certificate

To build authenticated key-exchange, we require more ingredients — namely, an integrity mechanism [e.g., a way to bind a message to a sender — a "public-key MAC" or digital signature]

We will revisit when discussing the TLS protocol

Digital signature scheme: Consists of three algorithms:

- Setup → (vk, sk): Outputs a verification key vk and a signing key sk
- Sign(sk, m) → σ: Takes the signing key sk and a message m and outputs a signature σ
- Verify(vk, m, σ) → 0/1: Takes the verification key vk, a message m, and a signature σ, and outputs a bit 0/1

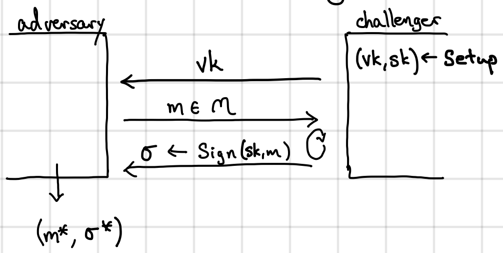
Two requirements:

- Correctness: For all messages $m \in \mathcal{M}$, $(vk, sk) \leftarrow \text{Setup}$, then

$$\Pr[\text{Verify}(vk, m, \text{Sign}(sk, m)) = 1] = 1.$$

[Honestly-generated signatures always verify]

- Unforgeability: Very similar to MAC security. For all efficient adversaries A, $\text{SigAdv}[A] = \Pr[W = 1] = \text{negl}(\lambda)$, where W is the output of the following experiment:


adversary
challenger
$(vk, sk) \leftarrow$ Setup
vk
$m \in \mathcal{M}$
$\sigma \leftarrow \text{Sign}(sk, m)$
(m^*, σ^*)

Let $m_1, \ldots, m_Q$ be the signing queries the adversary submits to the challenger Then, $W = 1$ if and only if:

$$\text{Verify}(vk, m^*, \sigma^*) = 1 \quad \text{and} \quad m^* \notin \{m_1, \ldots, m_Q\}$$

Adversary cannot produce a valid signature on a new message.

Exact analog of a MAC (slightly weaker unforgeability: require adversary to not be able to forge signature on new message)

↳ MAC security required that no forgery is possible on any message [needed for authenticated encryption]

digital signature algorithm

elliptic-curve DSA } standards (widely used on the web – eg, TLS)

It is possible to build digital signatures from discrete log based assumptions (DSA, ECDSA)

↳ But construction not intuitive until we see zero knowledge proofs

↳ We will first construct from RSA (trapdoor permutations)