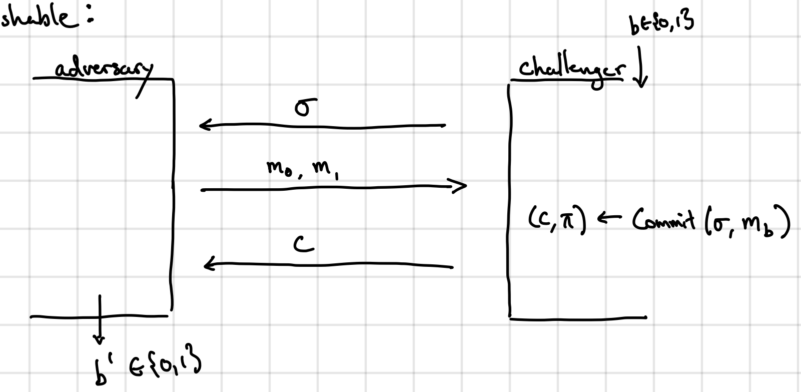
Requirements:

- Correctness: for all messages $m$:

$$\Pr[\sigma \leftarrow \text{Setup}\ ,\ (c,\pi) \leftarrow \text{Commit}(\sigma, m)\ ;\ \text{Verify}(\sigma, c, m, \pi) = 1] = 1$$

- Hiding: for all common reference strings $\sigma \in \{0,1\}^n$ and all efficient $A$, following distributions are computationally indistinguishable:

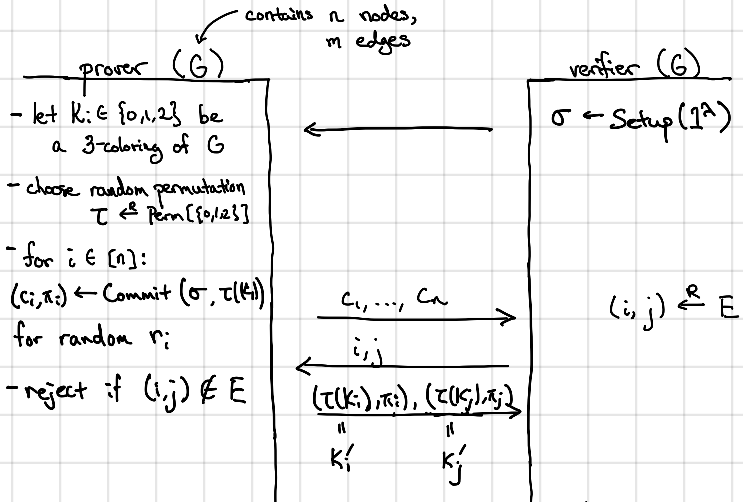


$$\left|\Pr[b' = 1 \mid b = 0] - \Pr[b' = 1 \mid b = 1]\right| = \text{negl}(\lambda)$$

- Binding: for all adversaries $A$, if $\sigma \leftarrow \text{Setup}$, then

$$\Pr[(m_0, m_1, c, \pi_0, \pi_1) \leftarrow A\ :\ m_0 \neq m_1 \text{ and } \text{Verify}(\sigma, c, m_0, \pi_0) = 1 = \text{Verify}(\sigma, c, m_1, \pi_1)] = \text{negl}$$

A ZK protocol for graph 3-coloring:


contains $n$ nodes, $m$ edges
prover $(G)$
verifier $(G)$
- let $k_i \in \{0,1,2\}$ be a 3-coloring of $G$
- choose random permutation $\tau \xleftarrow{R} \text{Perm}[\{0,1,2\}]$
- for $i \in [n]$:
$(c_i, \pi_i) \leftarrow \text{Commit}(\sigma, \tau(k_i))$
for random $r_i$
- reject if $(i,j) \notin E$
$\sigma \leftarrow \text{Setup}(1^\lambda)$
$c_1, \ldots, c_n$
$(i,j) \xleftarrow{R} E$
$i,j$
$(\underbrace{\tau(k_i)}_{k_i'}, \pi_i), (\underbrace{\tau(k_j)}_{k_j'}, \pi_j)$

→ accept if $k_i' \neq k_j'$ and $k_i', k_j' \in \{0,1,2\}$

$\text{Verify}(\sigma, c_i, k_i', \pi_i) = 1 = \text{Verify}(\sigma, c_j, k_j', \pi_j)$

reject otherwise

<u>Intuitively</u>: Prover commits to a coloring of the graph
Verifier challenges prover to reveal <u>coloring</u> of a single edge
Prover reveals the coloring on the chosen edge and opens the entries in the commitment

<u>Completeness</u>: By inspection [if coloring is valid, prover can always answer the challenge correctly]

<u>Soundness</u>: Suppose $G$ is <u>not</u> 3-colorable. Let $K_1, \dots, K_n$ be the except with prob. $1-\text{negl}$. coloring the prover committed to. If the commitment scheme is statistically binding, $c_1, \dots, c_n$ <u>uniquely</u> determine $K_1, \dots, K_n$. Since $G$ is not 3-colorable, there is an edge $(i,j) \in E$ where $K_i = K_j$ or $i \notin \{0,1,2\}$ or $j \notin \{0,1,2\}$. [Otherwise, $G$ is 3-colorable with coloring $K_1, \dots, K_n$.] Since the verifier chooses an edge to check at random, the verifier will choose $(i,j)$ with probability $1/|E|$. Thus, if $G$ is not 3-colorable,

$$\Pr[\text{verifier rejects}] \geq \frac{1}{|E|}$$

Thus, this protocol provides soundness $1 - \frac{1}{|E|}$. We can repeat this protocol $O(|E|^2)$ times <u>sequentially</u> to reduce soundness error to

$$\Pr[\text{verifier accepts proof of false statement}] \leq \left(1 - \frac{1}{|E|}\right)^{|E|^2} \leq e^{-|E|} = e^{-m} \quad \left[\text{since } 1 + x \leq e^x\right]$$

<u>Zero Knowledge</u>: We need to construct a simulator that outputs a valid transcript given only the graph $G$ as input. Let $V^*$ be a (possibly malicious) verifier. Construct simulator $S$ as follows:

1. Run $V^*$ to get $\sigma^*$.
2. Choose $K_i \leftarrow \{0,1,2\}$ for all $i \in [n]$.
   Let $(c_i, \pi_i) \leftarrow \text{Commit}(\sigma^*, K_i)$
   Give $(c_1, \dots, c_n)$ to $V^*$.
   } Simulator does <u>not</u> know coloring so it commits to a random one
3. $V^*$ outputs an edge $(i,j) \in E$
4. If $K_i \neq K_j$, then $S$ outputs $(K_i, K_j, \pi_i, \pi_j)$.
   Otherwise, restart and try again (if fails $\lambda$ times, then abort)

Simulator succeeds with probability $2/3$ (over choice of $K_1, \dots, K_n$). Thus, simulator produces a valid transcript with prob. $1 - \frac{1}{3^\lambda} = 1 - \text{negl}(\lambda)$ after $\lambda$ attempts. It suffices to show that simulated transcript is indistinguishable from a real transcript.

- <u>Real scheme</u>: prover opens $K_i, K_j$ where $K_i, K_j \xleftarrow{R} \{0,1,2\}$ [since prover randomly permutes the colors]
- <u>Simulation</u>: $K_i$ and $K_j$ sampled uniformly from $\{0,1,2\}$ and conditioned on $K_i \neq K_j$, distributions are identical

In addition, $(i,j)$ output by $V^*$ in the simulation is distributed correctly since commitment scheme is computationally-hiding (e.g. $V^*$ behaves essentially the same given commitments to a random coloring as it does given commitment to a valid coloring

If we repeat this protocol (for soundness amplification), simulator simulate one transcript at a time

<u>Summary</u>: Every language in NP has a zero-knowledge proof (assuming existence of PRGs)
↑
PRGs imply commitments