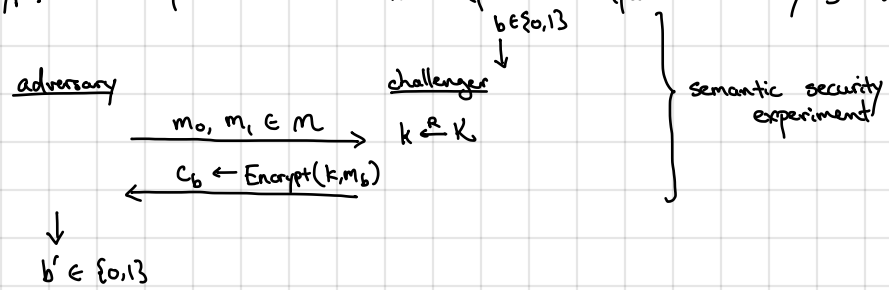
Now we will return to the notion of a secure encryption scheme:

Goal is to capture property that no efficient adversary can learn any information about the message given only the ciphertext. Suffices to argue that no efficient adversary can distinguish encryption of message m_0 from m_1, even if m_0, m_1 are adversarially-chosen.

Let (Encrypt, Decrypt) be a cipher. We define two experiments (parameterized by $b \in \{0,1\}$):



Adversary chooses two messages and receives encryption of one of them. Needs to guess which one (i.e., distinguish encryption of m_0 from encryption of m_1)

Let $W_0 := \Pr[b' = 1 \mid b = 0]$
$W_1 := \Pr[b' = 1 \mid b = 1]$
} probability that adversary guesses 1 (if adversary is good distinguisher, these two should be very different)

Define semantic security advantage of adversary A for cipher $\Pi_{SE} = (\text{Encrypt}, \text{Decrypt})$

$$\text{SSAdv}[A, \Pi_{SE}] = |W_0 - W_1|$$

Definition. A cipher $\Pi_{SE} = (\text{Encrypt}, \text{Decrypt})$ is semantically secure if for all efficient adversaries A,

$$\text{SSAdv}[A, \Pi_{SE}] = \text{negl}(\lambda)$$

↑ λ is a security parameter (here, models the bit-length of the key)

Understanding the definition:

Can we learn the least significant bit of a message given only the ciphertext (assuming a semantically-secure cipher)

No! Suppose we could. Then, adversary can choose two messages m_0, m_1 that differ in their least significant bit and distinguish with probability 1.

This generalizes to <u>any</u> efficiently-computable property of the two messages.

How does semantic security relate to perfect secrecy?

<u>Theorem</u>. If a cipher satisfies perfect secrecy, then it is semantically secure.

<u>Proof</u>. Perfect secrecy means that $\forall m_0, m_1 \in \mathcal{M}$, $c \in \mathcal{C}$:

$$\Pr[k \xleftarrow{R} K : \text{Encrypt}(k, m_0) = c] = \Pr[k \xleftarrow{R} K : \text{Encrypt}(k, m_1) = c]$$

Equivalently, the distributions

$$\underbrace{\{k \xleftarrow{R} K : \text{Encrypt}(k, m_0)\}}_{D_0} \quad \text{and} \quad \underbrace{\{k \xleftarrow{R} K : \text{Encrypt}(k, m_1)\}}_{D_1}$$

are <u>identical</u> ($D_0 \equiv D_1$). This means that the adversary's output b' is identically distributed in the two experiments, and so

$$SSAdv[A, \Pi_{SE}] = |W_0 - W_1| = 0.$$

<u>Corollary</u>. The one-time pad is semantically secure.

encryption key (PRG seed)

$c \leftarrow G(s) \oplus m$

$m \leftarrow G(s) \oplus c$

seems straightforward, but takes some care to prove

<u>Theorem</u>. Let G be a secure PRG. Then, the resulting stream cipher constructed from G is semantically secure.

<u>Proof</u>. Consider the semantic security experiments:

Experiment 0: Adversary chooses m_0, m_1 and receives $c_0 = G(s) \oplus m_0$

Experiment 1: Adversary chooses m_0, m_1 and receives $c_1 = G(s) \oplus m_1$

} Want to show that adversary's output in these two experiments are indistinguishable

Let $W_0 = \Pr[A \text{ outputs } 1 \text{ in Experiment } 0]$

$W_1 = \Pr[A \text{ outputs } 1 \text{ in Experiment } 1]$

<u>Idea</u>: If $G(s)$ is uniform random string (i.e., one-time pad), then $W_0 = W_1$. But $G(s)$ is like a one-time pad!

Define Experiment 0': Adversary chooses m_0, m_1 and receives $c_0 = t \oplus m_0$ where $t \xleftarrow{R} \{0,1\}^n$

Experiment 1': Adversary chooses m_0, m_1 and receives $c_1 = t \oplus m_1$ where $t \xleftarrow{R} \{0,1\}^n$

Define W_0', W_1' accordingly.

First, observe that $W_0' = W_1'$ (one-time pad is perfectly secure).

Now we show that $|W_0 - W_0'| = \text{negl}$ and $|W_1 - W_1'| < \text{negl}$.

$$\Rightarrow |W_0 - W_1| = |W_0 - W_0' + W_0' - W_1' + W_1' - W_1|$$

$$\leq |W_0 - W_0'| + |W_0' - W_1'| + |W_1' - W_1| \quad \text{by triangle inequality}$$

$$= \text{negl.} + \text{negl.} = \text{negl.}$$

Show. If G is a secure PRG, then for all efficient A, $|W_0 - W_0'| = \text{negl}$.

Common proof technique: prove the contrapositive.

Contrapositive: If A can distinguish Experiments 0 and 0', then G is not a secure PRG.

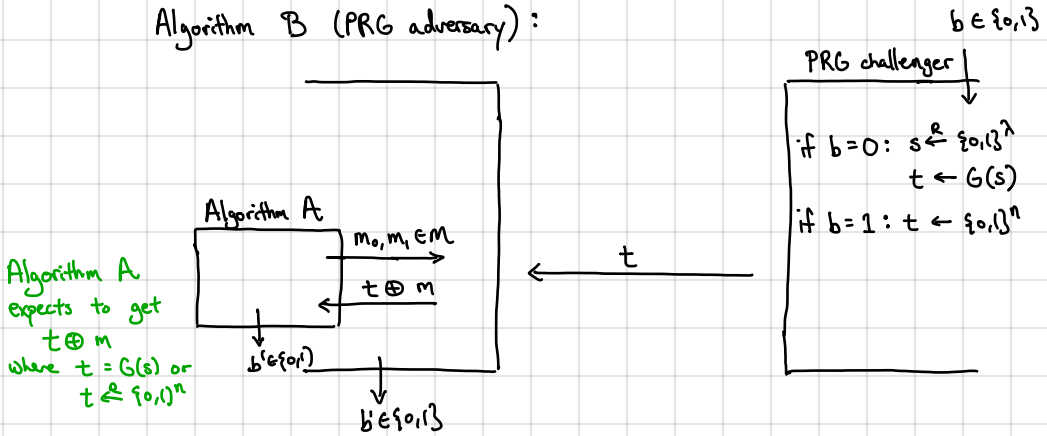
Suppose there exists efficient A that distinguishes Experiment 0 from 0'

$\Rightarrow$ We use A to construct efficient adversary B that breaks security of G.

↳ this step is a reduction

[we show how adversary (i.e., algorithm) for distinguishing Exp. 0 and 0' $\Rightarrow$ adversary for PRG]

Algorithm B (PRG adversary):



Running time of B = running time of A = efficient

Compute $\text{PRGAdv}[B, G]$.

$\Pr[B \text{ outputs } 1 \text{ if } b=0] = W_0$ ← if $b=0$, then A gets $G(s) \oplus m$ which is precisely the behavior in Exp. 0

$\Pr[B \text{ outputs } 1 \text{ if } b=1] = W_0'$ ← if $b=1$, then A gets $t \oplus m$ which is precisely the behavior in Exp. 0'

$\Rightarrow \text{PRGAdv}[B,G] = |W_0 - W_0'|$, which is non-negligible by assumption. This proves the contrapositive.

Important note: Security of above schemes shown assuming message space is $\{0,1\}^n$ (i.e., all messages are n-bits long)

In practice: We have variable-length messages. In this case, security guarantees indistinguishability from other messages of the same length, but length itself is leaked [inevitable if we want short ciphertexts]

↳ can be problematic — see traffic analysis attacks!

So far, we have shown that if we have a PRG, then we can encrypt messages efficiently (stream cipher)

Question: Do PRGs exist?

Unfortunately, we do not know!

Claim: If PRGs with non-trivial ($n > \lambda$) stretch exist, then $P \neq NP$.

Proof. Suppose $G: \{0,1\}^\lambda \to \{0,1\}^n$ is a secure PRG. Consider the following decision problem:

on input $t \in \{0,1\}^n$, does there exist $s \in \{0,1\}^\lambda$ such that $t = G(s)$

This problem is in NP (in particular, s is the witness). If G is secure, then no polynomial-time algorithm can solve this problem (if there was a polynomial-time algorithm for this problem, then it breaks PRF security with advantage $1 - \frac{1}{2^{n-\lambda}} > \frac{1}{2}$ since $n > \lambda$). Thus, $P \neq NP$.

In fact, there cannot even be a probabilistic polynomial-time algorithm that solves this problem with probability better than $\frac{1}{2} + \varepsilon$ for non-negligible $\varepsilon > 0$. This means that there is no BPP algorithm that breaks PRG security:

if PRGs exist, then $NP \not\subseteq BPP$

← bounded error probabilistic polynomial time

"randomized algorithms that solves problem with bounded (constant) error"

Thus, proving existence of PRG requires resolving long-standing open questions in complexity theory!

⇒ Cryptography: We will assume that certain problems are hard and base constructions of (hopefully small) number of conjectures.

- Hardness assumptions can be that certain mathematical problems are intractable (e.g., factoring)
  - ↳ typically for public-key cryptography (2nd half of this course)
- Hardness assumptions can be that certain constructions are secure (e.g., "AES is a secure block cipher")
  - ↳ typically for symmetric cryptography
  - ↳ constructions are more ad hoc, rely on heuristics, but very fast in practice

Examples of stream ciphers (PRGs): designed to be very fast (oftentimes with hardware support)

- Linear congruential generator (e.g., rand() function in C)

$r_{i+1} = a r_i + b \pmod m$

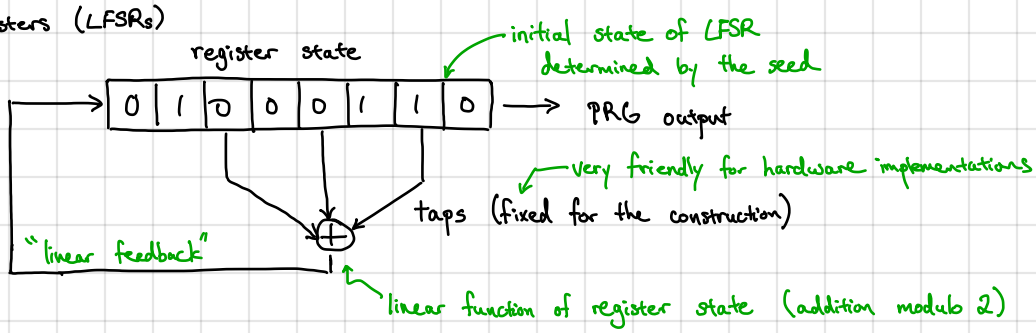
a, b, m are public constants
r_0 is the initial seed
} very simple, easy to implement (especially when m is a power of 2)
↳ need to choose so outputs have long period

→ typical implementation: output is a few bits of $r_0, r_1, r_2, \ldots$ (full value of $r_0, r_1, r_2, \ldots$ never revealed)
→ or $\lfloor r_i / w \rfloor$

Not a cryptographic PRG: NEVER USE rand() TO GENERATE CRYPTOGRAPHIC KEYS!

- Given full outputs, outputs fully predictable (if enough bits of state revealed, can brute force unknown bits)
- Even given partial outputs (e.g., least significant few bits of output) and having secret a, b, m, can still be broken (linear functions are not secure! see Boneh-Shoup Ch. 3.7.1 and related papers)
- Often good enough for non-cryptographic applications (e.g., statistical simulation)

- Linear feedback shift registers (LFSRs)



Each iteration: rightmost bit is output by LFSR
bits at tap positions are xored and shifted in from the left
1 clock cycle = 1 output bit — very simple and fast!

By itself, LFSR is totally broken: after observing n-bits of output, the entire state of the LFSR is known and subsequent bits are completely predictable!

Proposal: Use multiple LFSRs and combine in some non-linear way:

Example: CSS (content scrambling system) for DVD encryption [1996]

↳ actual CSS encryption has a few differences, but the core attack is unaffected

(needed to comply with export control restrictions)

C: carry bit from previous operation (initially 0)

- Brute-force attack: guess the seed ($\approx 2^{40}$ time)
- Can do much better with more clever strategy
  - ↳ General idea:
    - if we know a few bytes of output of the stream cipher and the output of the 17-bit LFSR, can subtract to obtain output of 25-bit LFSR
    - brute force the seed of the 17-bit LFSR, each guess induces a state for the 25-bit LFSR
    - check if output matches or not
  - ↳ Attack now runs in $\sim 2^{16}$ time
    - By 1999, full key-recovery attack on can recover key from DVD in just ~18 seconds on 450 MHz processor [totally broken!]

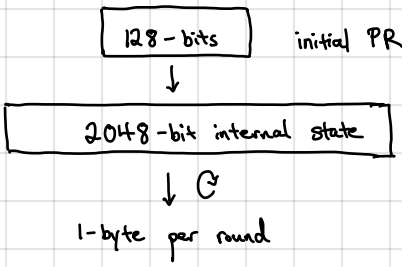
Pre-2000s:

Other examples: GSM encryption (A5/1,2 stream ciphers for encrypting GSM cell phone traffic)
- ↳ xor outputs of 3 LFSRs
- ↳ tried to keep cipher design private, but eventually reverse engineered and attacks found ← Snowden documents: NSA can process encrypted A5/1

Never rely on security by obscurity!

Bluetooth E0 stream cipher uses a design based on 4 LFSRs in conjuction with a 2-bit finite state machine — also not secure!

- RC4 stream cipher (1987) (widely used - SSL/TLS protocol, 802.11b)



Numerous problems:
- Bias in initial output: $\Pr[\text{second byte} = 0] = \frac{2}{256} > \frac{1}{256}$
  - ↳ When using RC4, recommendation is to ignore first 256 bytes due to potential bias
  - ↳ Correlations in output: probability of seeing (0,0) in output is $\frac{1}{256^2} + \frac{1}{256^3} > \frac{1}{256^2}$
  - ↳ Given outputs of RC4 with related keys (eg., keys sharing common suffix), possible to recover keys after seeing few blocks of output
    - ↳ Can be very problematic on weak devices (who may not have good sources of entropy)

- Modern stream ciphers (eSTREAM project: 2004-2008)
  - Salsa20 (2005) ⇝ ChaCha (2008)
    - ↳ core design maps 256-bit key, 64-bit nonce, 64-bit counter onto a 512-bit output

enables using same key (and different nonces) to encrypt multiple messages (will discuss later)

allows random access into the stream

Design is more complex:
- relies on a sequence of rounds
- each round consists of 32-bit additions, xors, and bit-shifts

  - ↳ very fast even in software (4-14 CPU cycles/output byte) — used to encrypt TLS traffic between Android and Google services