So far in this course: assumption is that adversary is <u>classical</u>.

How do things change if adversaries are <u>quantum</u>? We won't go into detail but will state main results:

<u>Grover's algorithm</u>: Given black-box access to a function $f : [N] \to \{0,1\}$, Grover's algorithm finds an $x \in [N]$ such that $f(x) = 1$ by making $O(\sqrt{N})$ queries to f.

"Searching an unsorted database of size N in time $O(\sqrt{N})$."

- <u>Classically</u>: Searching an unstructured database of size N requires time $\Omega(N)$ — cannot do better than a linear scan.
- <u>Quantum</u>: Grover's algorithm is tight for unstructured search. Any quantum algorithm for the unstructured search problem requires making $\Omega(\sqrt{N})$ queries (to the function / database).

$\Longrightarrow$ Quantum computes provide a <u>quadratic</u> speedup for unstructured search, and more broadly, function inversion.

<u>Implications in cryptography</u>: Consider a one-way function over a 128-bit domain. The task of inverting a one-way function is to find $x \in \{0,1\}^{128}$ such that $f(x) = y$ for some fixed target value f. Exhaustive search would take time $\approx 2^{128}$ on a classical computer, but using Grover's algorithm, can perform in time $\approx \sqrt{2^{128}} = 2^{64}$.

$\Longrightarrow$ For symmetric cryptography, need to <u>double</u> key-sizes to maintain same level of security (unless there are new quantum attacks on the underlying construction itself.

$\Longrightarrow$ Use AES-256 instead of AES-128 (<u>not</u> a significant change!)

Similar algorithm can be applied to obtain a quantum collision-finding algorithm that runs in time $\sqrt[3]{N}$ where N is the size of the domain (compare to $\sqrt{N}$ for the best classic algorithm)

↳ Instead of using SHA-256, use SHA-384 (<u>not</u> a significant change)

↳ The quantum algorithm require a large amount of space, so not clear that this is a significant threat, but even if it were, using hash functions with 384 bits of output suffices for security

<u>Main takeaway</u>: Symmetric cryptography mostly unaffected by quantum computers ~ generally just require a modest increase in key size

↳ e.g., symmetric encryption, MACs, authenticated encryption

Story more complicated for public-key primitives:

- Simon's algorithm and Shor's algorithm provide <u>polynomial-time</u> algorithms for solving discrete log (in any group with an efficiently-computable group operation) and for factoring
- Both algorithms rely on period finding (and more broadly, on solving the hidden subgroup problem)

Intuition for discrete log algorith (as a period finding problem):

- Let $(g, h = g^\alpha)$ be the discrete log instance in a group of prime order p
- Let $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{G}$ be the function
$$f(x,y) = g^x h^{-y}$$
- By construction,
$$f(x+\alpha, y+1) = g^{x+\alpha} h^{-y-1} = g^x h^{-y} g^\alpha h^{-1} = g^x h^{-y} = f(x,y)$$
- Thus, the element $(\alpha, -1)$ is the period of f, so using Shor's algorithm, we can efficient compute $(\alpha, -1)$ from (g, h), which yields the discrete log of h

Thus, if large scale quantum computers come online, we will need new cryptographic assumptions for our public-key primitives

↳ All the algebraic assumptions we have considered so far (e.g., discrete log, factoring) are broken

<u>How realistic is this threat?</u> - Lots of progress in building quantum computers recently by both academia and industry (e.g., see initiatives by Google, IBM, etc.)

- To run Shor's algorithm to factor a 2048-bit RSA modulus, estimated to need a quantum computer with ≈ 10000 <u>logical</u> qubits (analog of a bit in classical computers)
  - ↳ With quantum error correction, this requires ≥ 10 million <u>physical</u> qubits to realize
  - ↳ <u>Today</u>: machines with 10s of <u>physical</u> qubits, so still very far from being able to run Shor's algorithm
- Optimistic estimate: At least 20-30 years away

<u>Should we be concerned?</u> Quantum computers would break existing key-exchange and signature schemes

- <u>Signatures</u>: Future adversaries would be able to forge signatures under today's public keys, so if quantum computers come online, we can switch to and <u>only</u> use post-quantum schemes
- <u>Key-Exchange</u>: Future adversaries can break <u>confidentiality</u> of today's messages (i.e., we lose forward secrecy) — this is <u>problematic</u> in many scenarios (e.g., businesses want trade secrets to remain hidden for 50 years)

<u>This course</u>: will just focus on getting post-quantum signatures (will not discuss post-quantum key exchange)

[General approach for post-quantum cryptography: base hardness on assumptions believed to be hard on quantum computers (e.g., lattice-based cryptography, isogeny-based cryptography)]

For digital signatures, we will show that OWFs $\Rightarrow$ digital signatures

↳ Signatures can be based on <u>symmetric</u> primitives, so gives one approach to post-quantum signatures

<u>Lamport signatures</u>: Let $f: \mathcal{X} \to \mathcal{Y}$ be a one-way function (e.g., a PRG or CRHF)

length of message ($\mathcal{M} = \{0,1\}^n$)

- $\mathsf{Setup}(1^\lambda, 1^n)$: Sample $x_{i,b} \xleftarrow{R} \mathcal{X}$ $\forall i \in [n], b \in \{0,1\}$ and compute $y_{i,b} \leftarrow f(x_{i,b})$ $\forall i \in [2n], b \in \{0,1\}$
  Set

$$sk = \begin{array}{|c|c|c|c|} \hline x_{1,0} & x_{2,0} & \cdots & x_{n,0} \\ \hline x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\ \hline \end{array} \qquad pk = \begin{array}{|c|c|c|c|} \hline y_{1,0} & y_{2,0} & \cdots & y_{n,0} \\ \hline y_{1,1} & y_{2,1} & \cdots & y_{n,1} \\ \hline \end{array}$$

- $\mathsf{Sign}(sk, \underbrace{m}_{\in \{0,1\}^n})$: Output $(x_{1,m_1}, \ldots, x_{n,m_n})$
- $\mathsf{Verify}(vk, m, \sigma)$: Output 1 if $\forall i \in [n]$, $f(x_{i,m_i}) = y_{i,m_i}$ and 0 otherwise

<u>Theorem</u>. If f is one-way, then Lamport signatures are secure one-time signatures (i.e., where adversary can only make 1 signing query).

two signatures allow recovering secret key!

<u>Limitations</u>: One-time only [will fix later!]
Long public keys, secret keys, and signatures
- Compose with CRHF to get $\mathrm{poly}(\lambda)$-size parameters (independent of message length)
- Secret key can be derived from PRG (e.g., just λ bits)
- Public key can also be shortened to 2λ bits (special case of Winternitz construction below)

Many combinatoric tricks to reduce signature size

One-time signatures are very fast (only needs <u>symmetric cryptography</u>)
- Very useful in streaming setting: each packet in stream should be signed, but expensive to do so
  - Instead: include pk for one-time signature in first packet
    sign first packet using standard signature algorithm (public key)
    each packet includes OTS public key for next packet:

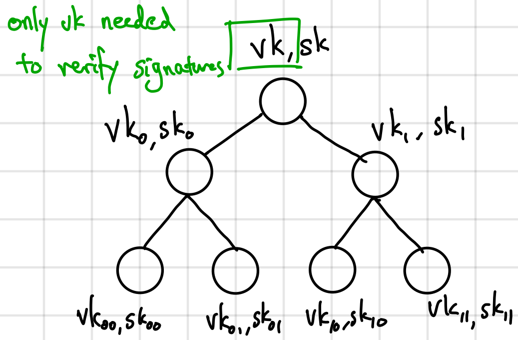
$$(m_0, vk_1), \sigma \to (m_1, vk_2), \sigma_1 \to (m_2, vk_3), \sigma_2, \ldots$$

signed using many-time signature — signed using secret key for vk_1 — signed using secret key for vk_2

chaining signatures

Stateful many-time signatures from one-time signatures:

Idea: use a tree of one-time signatures:



- every node is associated with a key-pair for an OTS scheme
- each signing key used to sign verification keys of its children
- signing key for leaf nodes used to sign messages
- each leaf can only be used to sign one message — need to keep track of which nodes have been used (stateful signature)

Example: Signing message m using (vk_{00}, sk_{00}):

- $\sigma_0 \leftarrow \text{Sign}(sk, vk_0 \| vk_1)$
- $\sigma_{00} \leftarrow \text{Sign}(sk_0, vk_{00} \| vk_{01})$
- $\sigma_m \leftarrow \text{Sign}(sk_{00}, m)$
- Output $(vk_0 \| vk_1, vk_{00} \| vk_{01}, \sigma_0, \sigma_{00}, \sigma_m)$

To verify, check

$\text{Verify}(vk, vk_0 \| vk_1, \sigma_0) \stackrel{?}{=} 1$

$\text{Verify}(vk_0, vk_{00} \| vk_{01}, \sigma_{00}) \stackrel{?}{=} 1$

$\text{Verify}(vk_{00}, m) \stackrel{?}{=} 1$

Only root vk needed here, all other keys included in σ

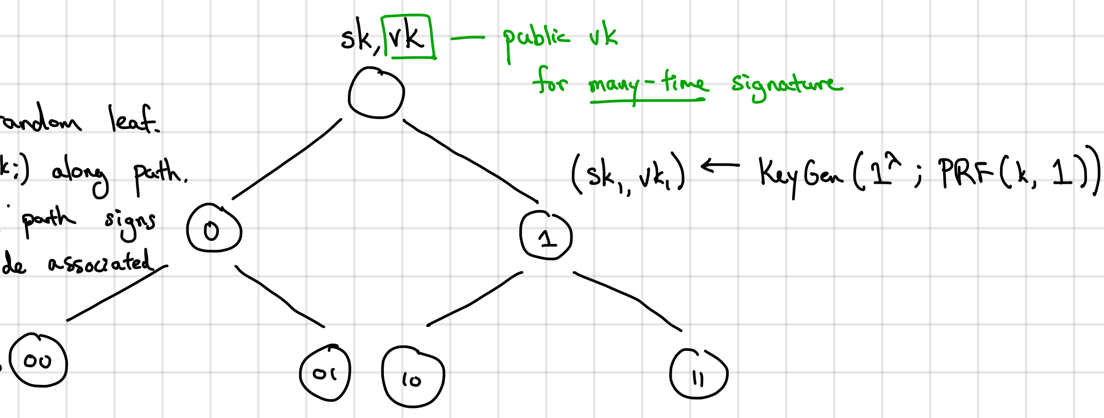
Security (Intuition):
- Keys for internal nodes only used to sign single message (verification keys of children)
- As long as leaf node never reused, then leaves are also only used once
- Security now reduces to one-time security of signature scheme

How to remove state?
- Consider a tree with 2^λ leaves and choose leaf at random for signing
- If we sign $\text{poly}(\lambda)$ messages, there will not be a collision in the leaf with $1 - \text{negl}(\lambda)$ probability
- Problem: Signing key is exponential (need to store $O(2^\lambda)$ signing keys)

Solution: Derive signing keys from a PRF!

$$(vk_i, sk_i) \leftarrow \text{KeyGen}(1^\lambda; \underbrace{\text{PRF}(k, i)}_{\text{randomness to key-generation algorithm}})$$

k: part of signing key; i: node index



To sign, choose random leaf.
Derive all (sk_i, vk_i) along path.
Each node along path signs verification node associated with children.
Leaf node signs message.

$(sk_{10}, vk_{10}) \leftarrow \text{KeyGen}(1^\lambda; \text{PRF}(k, 10))$

Signature contains complete validation path from root to leaf and signature of leaf on message.
Every internal node still signs only one message.