Question: Do PRGs exist?

Unfortunately, we do not know!

Claim: If PRGs with non-trivial ($n > \lambda$) stretch exist, then $P \neq NP$.

Proof. Suppose $G: \{0,1\}^\lambda \to \{0,1\}^n$ is a secure PRG. Consider the following decision problem:

on input $t \in \{0,1\}^n$, does there exist $s \in \{0,1\}^\lambda$ such that $t = G(s)$

This problem is in NP (in particular, s is the witness). If G is secure, then no polynomial-time algorithm can solve this problem (if there was a polynomial-time algorithm for this problem, then it breaks PRF security with advantage $1 - \frac{1}{2^{n-\lambda}} > \frac{1}{2}$ since $n > \lambda$). Thus, $P \neq NP$.

In fact, there cannot even be a probabilistic polynomial-time algorithm that solves this problem with probability better than $\frac{1}{2} + \varepsilon$ for non-negligible $\varepsilon > 0$. This means that there is no BPP algorithm that breaks PRG security:

if PRGs exist, then $NP \not\subseteq BPP$

← bounded error probabilistic polynomial time

"randomized algorithms that solves problem with bounded (constant) error"

Thus, proving existence of PRG requires resolving long-standing open questions in complexity theory!

⇒ Cryptography: We will assume that certain problems are hard and base constructions of (hopefully small) number of conjectures.

- Hardness assumptions can be that certain mathematical problems are intractable (e.g., factoring)
  - ↳ typically for public-key cryptography (2nd half of this course)
- Hardness assumptions can be that certain constructions are secure (e.g., "AES is a secure block cipher")
  - ↳ typically for symmetric cryptography
  - ↳ constructions are more ad hoc, rely on heuristics, but very fast in practice

Examples of stream ciphers (PRGs): designed to be very fast (oftentimes with hardware support)

- Linear congruential generator (e.g., rand() function in C)

$r_{i+1} = a r_i + b \pmod m$

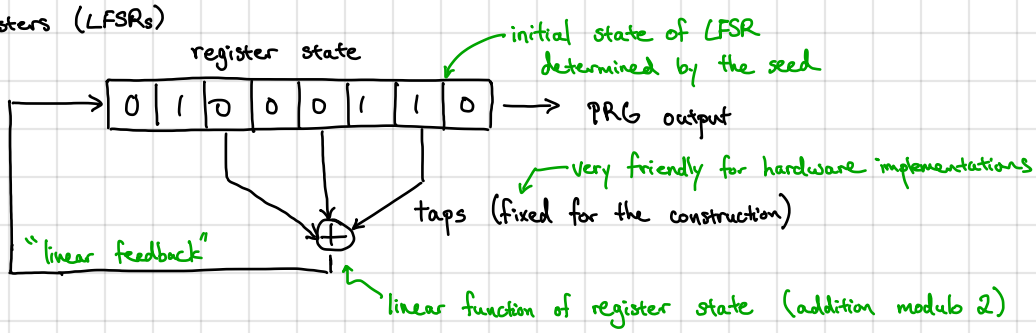
a, b, m are public constants
r_0 is the initial seed
} very simple, easy to implement (especially when m is a power of 2)
↳ need to choose so outputs have long period

→ typical implementation: output is a few bits of $r_0, r_1, r_2, \ldots$ (full value of $r_0, r_1, r_2, \ldots$ never revealed)
→ or $\lfloor r_i / w \rfloor$

Not a cryptographic PRG: NEVER USE rand() TO GENERATE CRYPTOGRAPHIC KEYS!

- Given full outputs, outputs fully predictable (if enough bits of state revealed, can brute force unknown bits)
- Even given partial outputs (e.g., least significant few bits of output) and having secret a, b, m, can still be broken (linear functions are not secure! see Boneh-Shoup Ch. 3.7.1 and related papers)
- Often good enough for non-cryptographic applications (e.g., statistical simulation)

- Linear feedback shift registers (LFSRs)



Each iteration: rightmost bit is output by LFSR
bits at tap positions are xored and shifted in from the left
1 clock cycle = 1 output bit — very simple and fast!

By itself, LFSR is totally broken: after observing n-bits of output, the entire state of the LFSR is known and subsequent bits are completely predictable!

Proposal: Use multiple LFSRs and combine in some non-linear way:

Example: CSS (content scrambling system) for DVD encryption [1996]

↳ actual CSS encryption has a few differences, but the core attack is unaffected

(needed to comply with export control restrictions)

C: carry bit from previous operation (initially 0)

- Brute-force attack: guess the seed ($\approx 2^{40}$ time)
- Can do much better with more clever strategy
  - ↳ General idea:
    - if we know a few bytes of output of the stream cipher and the output of the 17-bit LFSR, can subtract to obtain output of 25-bit LFSR
    - brute force the seed of the 17-bit LFSR, each guess induces a state for the 25-bit LFSR
    - check if output matches or not
  - ↳ Attack now runs in $\sim 2^{16}$ time
    - By 1999, full key-recovery attack on can recover key from DVD in just ~18 seconds on 450 MHz processor [totally broken!]

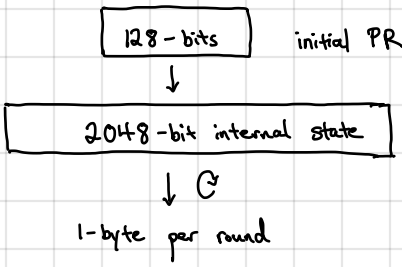
Pre-2000s:

Other examples: GSM encryption (A5/1,2 stream ciphers for encrypting GSM cell phone traffic)
- ↳ xor outputs of 3 LFSRs
- ↳ tried to keep cipher design private, but eventually reverse engineered and attacks found ← Snowden documents: NSA can process encrypted A5/1

Never rely on security by obscurity!

Bluetooth E0 stream cipher uses a design based on 4 LFSRs in conjuction with a 2-bit finite state machine — also not secure!

- RC4 stream cipher (1987) (widely used - SSL/TLS protocol, 802.11b)



Numerous problems:
- Bias in initial output: $\Pr[\text{second byte} = 0] = \frac{2}{256} > \frac{1}{256}$
  - ↳ When using RC4, recommendation is to ignore first 256 bytes due to potential bias
  - ↳ Correlations in output: probability of seeing (0,0) in output is $\frac{1}{256^2} + \frac{1}{256^3} > \frac{1}{256^2}$
  - ↳ Given outputs of RC4 with related keys (eg., keys sharing common suffix), possible to recover keys after seeing few blocks of output
    - ↳ Can be very problematic on weak devices (who may not have good sources of entropy)

- Modern stream ciphers (eSTREAM project: 2004-2008)
  - Salsa20 (2005) ⇝ ChaCha (2008)
    - ↳ core design maps 256-bit key, 64-bit nonce, 64-bit counter onto a 512-bit output

enables using same key (and different nonces) to encrypt multiple messages (will discuss later)

allows random access into the stream

Design is more complex:
- relies on a sequence of rounds
- each round consists of 32-bit additions, xors, and bit-shifts

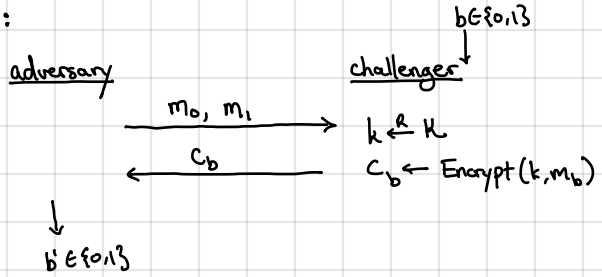
  - ↳ very fast even in software (4-14 CPU cycles/output byte) — used to encrypt TLS traffic between Android and Google services

Recall: the one-time pad is not reusable (i.e., the two-time pad is totally broken)

NEVER REUSE THE KEY TO A STREAM CIPHER!

But wait... we "proved" that a stream cipher was secure, and yet, there is an attack?

Recall security game:



Observe: adversary only sees one ciphertext

key is only used once

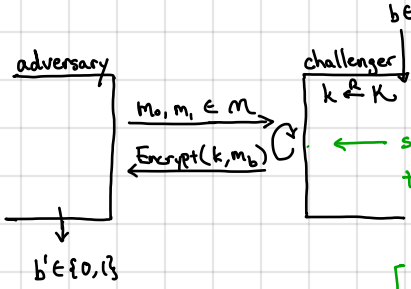
⇒ Security in this model says nothing about multiple messages / ciphertexts

Problem: If we want security with multiple ciphertexts, we need a different or stronger definition (CPA security)

Definition: An encryption scheme $\Pi_{SE} = (\text{Encrypt}, \text{Decrypt})$ is secure against chosen-plaintext attacks (CPA-secure) if for all efficient adversaries A:

$$\text{CPAAdv}[A, \Pi_{SE}] = \left| \Pr[W_0 = 1] - \Pr[W_1 = 1] \right| = \text{negl}.$$

where W_b ($b \in \{0,1\}$) is the output of the following experiment:



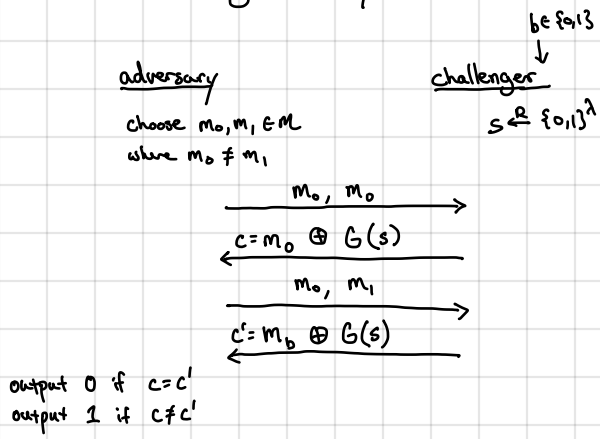
← same idea as in original semantic security game, but allow adversary to make encryption queries (also called a "left-or-right" oracle)

output of experiment W_b

[Adversary's goal is to guess which of m_0 or m_1 was encrypted, given access to an encryption oracle (i.e., adversary gets to see encryptions of messages of its choice.]

Claim. A stream cipher is not CPA-secure.

Proof. Consider the following adversary:



$\Pr[b' = 1 \mid b = 0] = 0$ since $c' = m_0 \oplus G(s) = c$

$\Pr[b' = 1 \mid b = 1] = 1$ since $c' = m_1 \oplus G(s) \neq c$

$\Rightarrow \text{CPAAdv}[A, \Pi_{SE}] = 1$

Observe: Above attack works for any deterministic encryption scheme.

$\Rightarrow$ CPA-secure encryption must be randomized!

$\Rightarrow$ To be reusable, cannot be deterministic. Encrypting the same message twice should not reveal that identical messages were encrypted.

To build a CPA-secure encryption scheme, we will use a "block cipher"

- Block cipher is an invertible keyed function that takes a block of n input bits and produces a block of n output bits
- Examples include 3DES (key size 168 bits, block size 64 bits)
  AES (key size 128 bits, block size 128 bits)

Will define block ciphers abstractly first: pseudorandom functions (PRFs) and pseudorandom permutations (PRPs) (block ciphers)

↳ General idea: PRFs behave like random functions
PRPs behave like random permutations