Lattice-Based Functional Commitments: Constructions and Cryptanalysis

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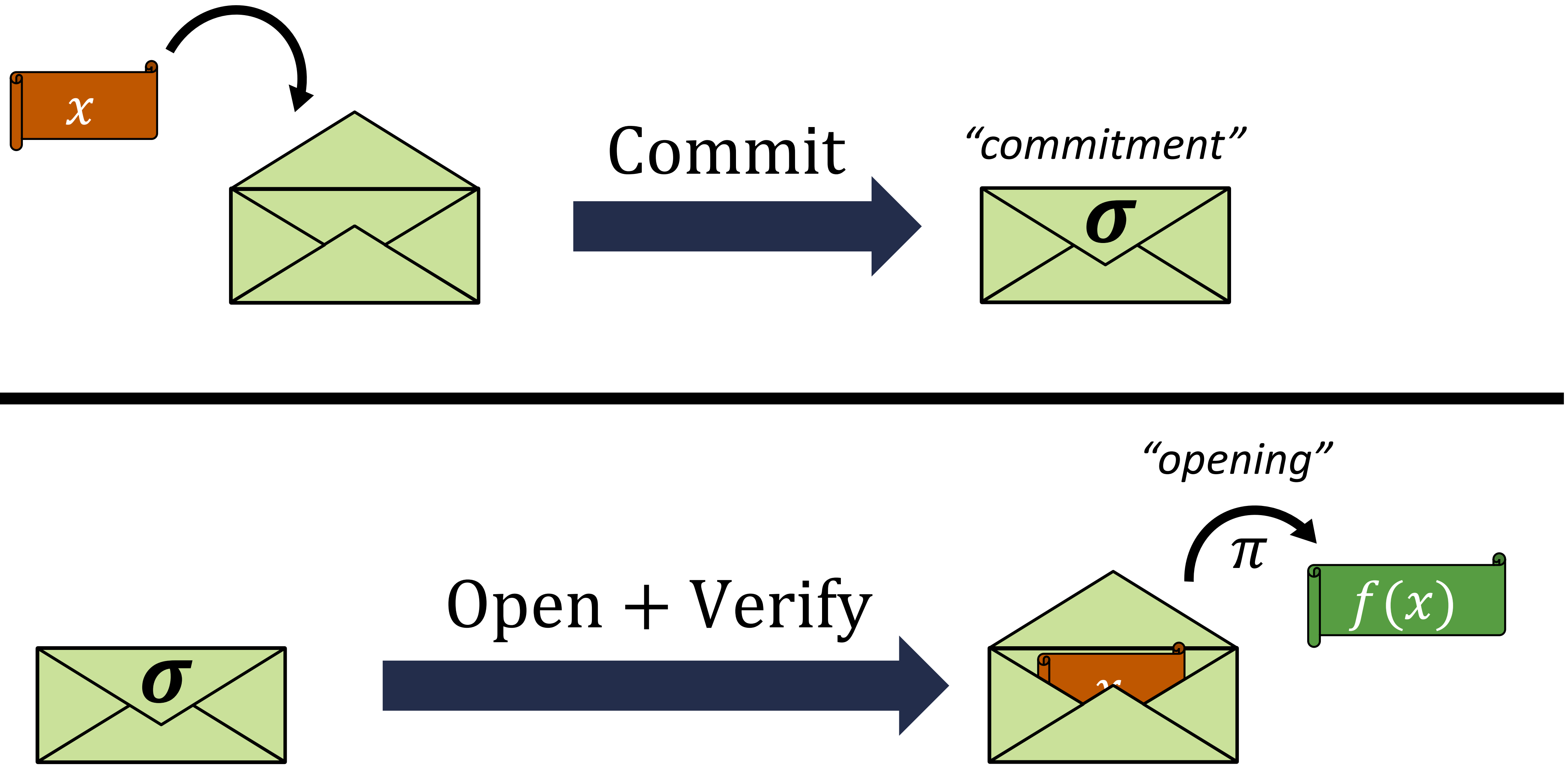
based on joint works with Hoeteck Wee

Functional Commitments



Functional Commitments

$$\mathrm{Commit}(\mathrm{crs}, x) \to (\sigma, \mathrm{st})$$

Takes a common reference string and commits to an input x

Outputs commitment σ and commitment state st

Functional Commitments

$$\mathrm{Commit}(\mathrm{crs}, x) \to (\sigma, \mathrm{st})$$

$$\mathrm{Open}(\mathrm{st}, f) \to \pi$$

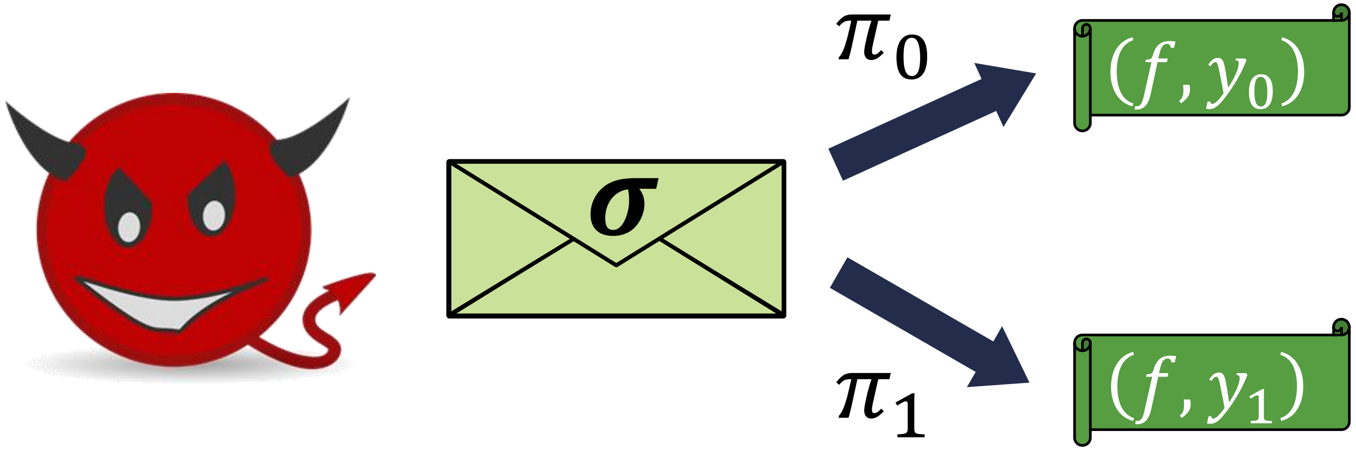
Takes the commitment state and a function f and outputs an opening π

$$\mathrm{Verify}(\mathrm{crs}, \sigma, (f, y), \pi) \to 0/1$$

Checks whether π is valid opening of σ to value y with respect to f

Functional Commitments

Binding: efficient adversary cannot open σ to two different values with respect to the **same** f



$\mathrm{Verify}(\mathrm{crs}, \sigma, (f, y_0), \pi_0) = 1$

$\mathrm{Verify}(\mathrm{crs}, \sigma, (f, y_1), \pi_1) = 1$

Functional Commitments

Succinctness: commitments and openings should be short

- **Short commitment:** $|\sigma| = \mathrm{poly}(\lambda, \log|x|)$
- **Short opening:** $|\pi| = \mathrm{poly}(\lambda, \log|x|)$

Will consider relaxation where $|\sigma|$ and $|\pi|$ can grow with **depth** of the circuit computing f

Special Cases of Functional Commitments

Vector commitments:

$[x_1, x_2, \ldots, x_n]$ $\xrightarrow{\mathrm{ind}_i(x_1, \ldots, x_n) \coloneqq x_i}$ x_i

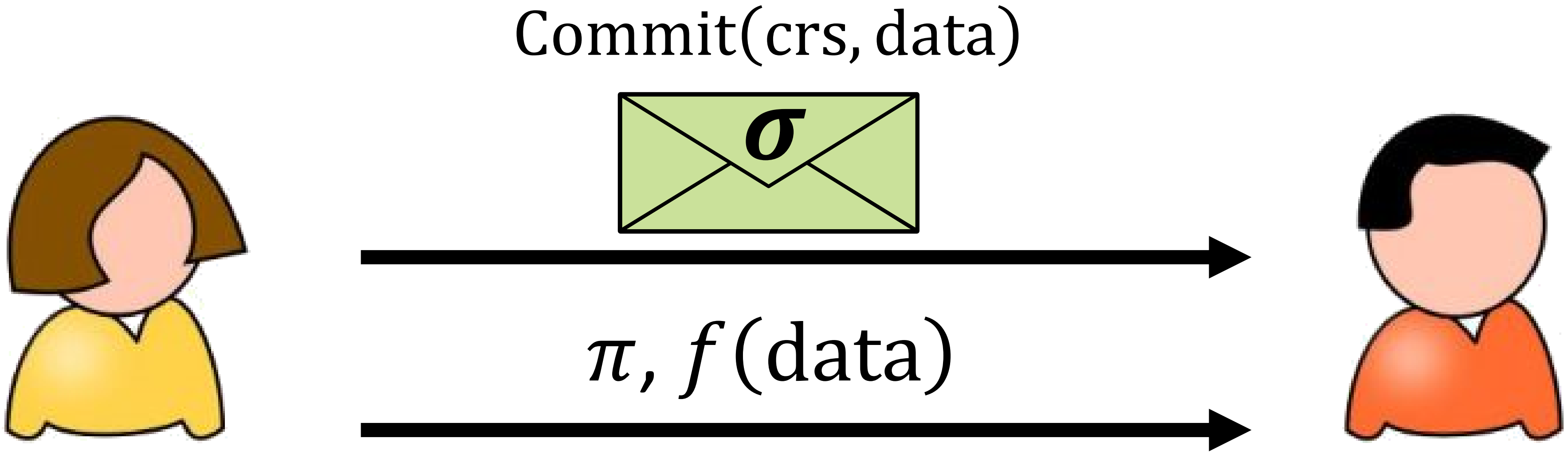
commit to a vector, open at an index

Polynomial commitments:

$[\alpha_0, \alpha_1, \ldots, \alpha_d]$ $\xrightarrow{f_x(\alpha_0, \ldots, \alpha_d) \coloneqq \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d}$ y

commit to a polynomial, open to the evaluation at x

Commitments as Proofs on Committed Data



π is a proof that the data satisfies some property
(e.g., committed input is in a certain range)

Succinctness: both the commitment and the proof are short

Succinct Functional Commitments

(not an exhaustive list!)

Scheme	Function Class	Assumption
[Mer87]	vector commitment	collision-resistant hash functions
[LY10, CF13, LM19, GRWZ20]	vector commitment	q-type pairing assumptions
[CF13, LM19, BBF19]	vector commitment	groups of unknown order
[PPS21]	vector commitment	short integer solutions (SIS)
[KZG10, Lee20]	polynomial commitment	q-type pairing assumptions
[BFS19, BHRRS21, BF23]	polynomial commitment	groups of unknown order
[LRY16]	linear functions	q-type pairing assumptions
[ACLMT22]	constant-degree polynomials	k-R-ISIS assumption (falsifiable)
[LRY16]	Boolean circuits	collision-resistant hash functions + SNARKs
[dCP23]	Boolean circuits	SIS (non-succinct openings in general)
[KLVW23]	Boolean circuits	LWE (via batch arguments)
[BCFL23]	Boolean circuits	twin k-R-ISIS (or q-type pairing assumption)
[WW23a, WW23b]	Boolean circuits	ℓ-succinct SIS **This talk**
[WW24]	Boolean circuits	k-Lin (pairings)

Framework for Lattice Commitments

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Common reference string (for inputs of length ℓ):

matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_\ell \in \mathbb{Z}_q^{n \times m}$

target vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_\ell \in \mathbb{Z}_q^n$

auxiliary data: cross-terms $\boldsymbol{u}_{ij} \leftarrow \boldsymbol{A}_i^{-1}(\boldsymbol{t}_j) \in \mathbb{Z}_q^m$ where $i \neq j$

short (i.e., low-norm) vector satisfying $\boldsymbol{A}_i \boldsymbol{u}_{ij} = \boldsymbol{t}_j$

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Commitment to $\boldsymbol{x} \in \mathbb{Z}_q^\ell$:

$$\boldsymbol{c} = \sum_{i \in [\ell]} x_i \boldsymbol{t}_i$$

linear combination of target vectors

Opening to value y at index i:

short $\boldsymbol{v}_i$ such that $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + y \cdot \boldsymbol{t}_i$

Honest opening:

Correct as long as $\boldsymbol{x}$ is short

$$\boldsymbol{v}_i = \sum_{j \neq i} x_j \boldsymbol{u}_{ij}$$

$$\boxed{\boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i = \sum_{j \neq i} x_j \boldsymbol{A}_i \boldsymbol{u}_{ij} + x_i \boldsymbol{t}_i = \sum_{j \in [\ell]} x_j \boldsymbol{t}_j = \boldsymbol{c}}$$

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auxiliary data: cross-terms $\boldsymbol{u}_{ij} \leftarrow \boldsymbol{A}_i^{-1}(\boldsymbol{t}_j) \in \mathbb{Z}_q^m$ where $i \neq j$

[PPS21]: $\boldsymbol{A}_i \leftarrow \mathbb{Z}_q^{n \times m}$ and $\boldsymbol{t}_i \leftarrow \mathbb{Z}_q^n$ are independent and uniform

suffices for vector commitments (from SIS)

[ACLMT21]: $\boldsymbol{A}_i = \boldsymbol{W}_i \boldsymbol{A}$ and $\boldsymbol{t}_i = \boldsymbol{W}_i \boldsymbol{u}_i$ where $\boldsymbol{W}_i \leftarrow \mathbb{Z}_q^{n \times n}, \boldsymbol{A} \leftarrow \mathbb{Z}_q^{n \times m}, \boldsymbol{u}_i \leftarrow \mathbb{Z}_q^n$

(one candidate adaptation to the integer case)

<u>generalizes</u> to functional commitments for constant-degree polynomials (from k-R-ISIS)

Our Approach

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

for a short $\boldsymbol{v}_i$

Our approach: rewrite ℓ relations as a single linear system

$$\left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{I}_n \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{I}_n \end{array}\right] \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\ell \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} -x_1 \boldsymbol{t}_1 \\ \vdots \\ -x_\ell \boldsymbol{t}_\ell \end{bmatrix}$$

$\boldsymbol{I}_n$ denotes the identity matrix

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Our approach: rewrite ℓ relations as a single linear system

$$\left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right] \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\ell \\ \hat{\boldsymbol{c}} \end{bmatrix} = \begin{bmatrix} -x_1 \boldsymbol{t}_1 \\ \vdots \\ -x_\ell \boldsymbol{t}_\ell \end{bmatrix}$$

For security and functionality, it will be useful to write $\boldsymbol{c} = \boldsymbol{G}\hat{\boldsymbol{c}}$

"powers of two matrix"

$$\boldsymbol{G} = \begin{bmatrix} 1 & 2 & \cdots & 2^{\lfloor \log q \rfloor} & & & & \\ & & & & \ddots & & & \\ & & & & 1 & 2 & \cdots & 2^{\lfloor \log q \rfloor} \end{bmatrix}$$

Our Approach

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- target vectors $\boldsymbol{t}_1, \dots, \boldsymbol{t}_\ell \in \mathbb{Z}_q^n$
- *auxiliary data:* cross-terms $\boldsymbol{u}_{ij} \leftarrow \boldsymbol{A}_i^{-1}(\boldsymbol{t}_j)$

Our Approach

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

for a short $\boldsymbol{v}_i$

Our approach: rewrite ℓ relations as a single linear system (and publish a trapdoor for it)

$$\underbrace{\left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right]}_{\boldsymbol{B}_\ell} \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\ell \\ \hat{\boldsymbol{c}} \end{bmatrix} = \begin{bmatrix} -x_1 \boldsymbol{t}_1 \\ \vdots \\ -x_\ell \boldsymbol{t}_\ell \end{bmatrix}$$

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auxiliary data: ~~cross-terms $\boldsymbol{u}_{ij} \leftarrow \boldsymbol{A}_i^{-1}(\boldsymbol{t}_j)$~~

trapdoor for $\boldsymbol{B}_\ell$

Trapdoor for $\boldsymbol{B}_\ell$ can be used to sample short solutions $\boldsymbol{x}$ to the linear system $\boldsymbol{B}_\ell \boldsymbol{x} = \boldsymbol{y}$ (for arbitrary $\boldsymbol{y}$)

Our Approach

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

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Committing to an input $\boldsymbol{x}$:

Use trapdoor for $\boldsymbol{B}_\ell$ to **jointly** sample a solution $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \hat{\boldsymbol{c}}$

$\boldsymbol{c} = \boldsymbol{G}\hat{\boldsymbol{c}}$ is the commitment and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$ are the openings

Proving Security

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

for a short $\boldsymbol{v}_i$

Suppose adversary can break binding

outputs $\boldsymbol{c}, (\boldsymbol{v}_i, x_i), (\boldsymbol{v}_i', x_i')$ such that

$$\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i$$

$$= \boldsymbol{A}_i \boldsymbol{v}_i' + x_i' \boldsymbol{t}_i$$

set $\boldsymbol{A}_i \leftarrow \mathbb{Z}_q^{n \times m}$

set $\boldsymbol{t}_i = \boldsymbol{e}_1 = [1,0,\ldots,0]^{\mathrm{T}}$

(*cannot set* $\boldsymbol{t}_i = \boldsymbol{0}$ *as otherwise, it could be* $\boldsymbol{v}_i = \boldsymbol{v}_i'$)

Short integer solutions (SIS)

given $\boldsymbol{A} \leftarrow \mathbb{Z}_q^{n \times m}$, hard to find short $\boldsymbol{x} \neq 0$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

$$\boldsymbol{A}_i \underbrace{(\boldsymbol{v}_i - \boldsymbol{v}_i')}_{(short)} = \underbrace{(x_i' - x_i)}_{(non\text{-}zero)} \boldsymbol{t}_i$$

Looks like an SIS solution…

How to choose $\boldsymbol{A}_i, \boldsymbol{t}_i$?

Proving Security

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

for a short $\boldsymbol{v}_i$

Suppose adversary can break binding

outputs $\boldsymbol{c}, (\boldsymbol{v}_i, x_i), (\boldsymbol{v}_i', x_i')$ such that

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \\ &= \boldsymbol{A}_i \boldsymbol{v}_i' + x_i' \boldsymbol{t}_i \end{aligned}$$

set $\boldsymbol{A}_i \leftarrow \mathbb{Z}_q^{n \times m}$

set $\boldsymbol{t}_i = \boldsymbol{e}_1 = [1,0,\dots,0]^{\mathrm{T}}$

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Short integer solutions (SIS)

given $\boldsymbol{A} \leftarrow \mathbb{Z}_q^{n \times m}$, hard to find short $\boldsymbol{x} \neq 0$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

$$\boldsymbol{A}_i(\boldsymbol{v}_i - \boldsymbol{v}_i') = (x_i' - x_i)\boldsymbol{e}_1$$

$\boldsymbol{v}_i - \boldsymbol{v}_i'$ is a SIS solution for $\boldsymbol{A}_i$ without the first row

Proving Security

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

for a short $\boldsymbol{v}_i$

Adversary that breaks binding can solve SIS with respect to $\boldsymbol{A}_i$

(technically $\boldsymbol{A}_i$ without the first row – which is equivalent to SIS with dimension $n-1$)

but… adversary also gets additional information beyond $\boldsymbol{A}_i$

$$\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right]$$

Adversary sees **trapdoor** for $\boldsymbol{B}_\ell$

Basis-Augmented SIS (BASIS) Assumption

Captures and generalizes other lattice-based functional commitments [PPS21, ACLMT22]

Verification invariant: $\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$

for a short $\boldsymbol{v}_i$

Adversary that breaks binding can solve SIS with respect to $\boldsymbol{A}_i$

Basis-augmented SIS (BASIS) assumption:

SIS is hard with respect to $\boldsymbol{A}_i$
given a trapdoor (a basis) for the matrix

$$\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right]$$

Can simulate CRS from BASIS challenge:

matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_\ell \leftarrow \mathbb{Z}_q^{n \times m}$

trapdoor for $\boldsymbol{B}_\ell$

Basis-Augmented SIS (BASIS) Assumption

SIS is hard with respect to $\boldsymbol{A}_i$ given a trapdoor (a basis) for the matrix

$$\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right]$$

When $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_\ell \leftarrow \mathbb{Z}_q^{n \times m}$ are uniform and independent:

hardness of SIS implies hardness of BASIS

(follows from standard lattice trapdoor extension techniques)

Vector Commitments from SIS

Common reference string (for inputs of length ℓ):

matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_\ell \in \mathbb{Z}_q^{n \times m}$

auxiliary data: trapdoor for $\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right]$

To commit to a vector $\boldsymbol{x} \in \mathbb{Z}_q^\ell$: sample solution $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell, \hat{\boldsymbol{c}})$

$$\left[\begin{array}{ccc|c} \boldsymbol{A}_1 & & & -\boldsymbol{G} \\ & \ddots & & \vdots \\ & & \boldsymbol{A}_\ell & -\boldsymbol{G} \end{array}\right] \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\ell \\ \hat{\boldsymbol{c}} \end{bmatrix} = \begin{bmatrix} -x_1 \boldsymbol{e}_1 \\ \vdots \\ -x_\ell \boldsymbol{e}_\ell \end{bmatrix}$$

Commitment is $\boldsymbol{c} = \boldsymbol{G}\hat{\boldsymbol{c}}$ Openings are $\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell$

Can commit and open to **arbitrary** $\mathbb{Z}_q$ vectors

Commitments and openings statistically **hide** unopened components

Linearly homomorphic:
$\boldsymbol{c} + \boldsymbol{c}'$ is a commitment to $\boldsymbol{x} + \boldsymbol{x}'$ with openings $\boldsymbol{v}_i + \boldsymbol{v}_i'$

Extending to Functional Commitments

Goal: commit to $\boldsymbol{x} \in \{0,1\}^{\ell}$, open to function $f(\boldsymbol{x})$

Suppose $f(\boldsymbol{x}) = \sum_{i \in [\ell]} \alpha_i x_i$ is a **linear** function

$$\textbf{Verification invariant: } \boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i \qquad \forall i \in [\ell]$$

Can also view $\boldsymbol{c}$ as commitment to vector $x_i \boldsymbol{t}_i$ with respect to $\boldsymbol{A}_i$ and opening $\boldsymbol{v}_i$

Suppose $\boldsymbol{c}_1, \boldsymbol{c}_2$ are commitments to vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ with respect to the same $\boldsymbol{A}$

$$\begin{aligned} \boldsymbol{c}_1 &= \boldsymbol{A}\boldsymbol{v}_1 + \boldsymbol{u}_1 \\ \boldsymbol{c}_2 &= \boldsymbol{A}\boldsymbol{v}_2 + \boldsymbol{u}_2 \end{aligned} \quad \Longrightarrow \quad \boldsymbol{c}_1 + \boldsymbol{c}_2 = \boldsymbol{A}(\boldsymbol{v}_1 + \boldsymbol{v}_2) + (\boldsymbol{u}_1 + \boldsymbol{u}_2)$$

Extending to Functional Commitments

$$\boxed{\begin{aligned} \boldsymbol{c}_1 &= \boldsymbol{A}\boldsymbol{v}_1 + x_1\boldsymbol{t} \\ &\vdots \\ \boldsymbol{c}_\ell &= \boldsymbol{A}\boldsymbol{v}_\ell + x_\ell\boldsymbol{t} \end{aligned}}$$

Desired correctness relation

$$\Downarrow$$

$$\boxed{\begin{aligned} \boldsymbol{W}_1\boldsymbol{c} &= \boldsymbol{A}\boldsymbol{v}_1 + x_1\boldsymbol{t} \\ &\vdots \\ \boldsymbol{W}_\ell\boldsymbol{c} &= \boldsymbol{A}\boldsymbol{v}_\ell + x_\ell\boldsymbol{t} \end{aligned}}$$

Cannot define commitment to be $(\boldsymbol{c}_1, \dots, \boldsymbol{c}_\ell)$ since this is long

Instead, suppose $\boldsymbol{c}_i = \boldsymbol{W}_i\boldsymbol{c}$ can be **derived** from a (single) $\boldsymbol{c}$

$$\overbrace{\left[\begin{array}{ccc|c} \boldsymbol{A} & & & -\boldsymbol{W}_1 \\ & \ddots & & \vdots \\ & & \boldsymbol{A} & -\boldsymbol{W}_\ell \end{array}\right]}^{\boldsymbol{B}_\ell} \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\ell \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} -x_1\boldsymbol{t} \\ \vdots \\ -x_\ell\boldsymbol{t} \end{bmatrix}$$

Our approach: rewrite ℓ relations as a single linear system (and publish a trapdoor for it)

Extending to Functional Commitments

$$\underbrace{\left[\begin{array}{ccc:c} \boldsymbol{A} & & & -\boldsymbol{W}_1 \\ & \ddots & & \vdots \\ & & \boldsymbol{A} & -\boldsymbol{W}_\ell \end{array}\right]}_{\boldsymbol{B}_\ell} \cdot \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_\ell \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} -x_1 \boldsymbol{t} \\ \vdots \\ -x_\ell \boldsymbol{t} \end{bmatrix}$$

CRS contains $\boldsymbol{A}, \boldsymbol{W}_1, \dots, \boldsymbol{W}_\ell, \boldsymbol{t}$ and trapdoor for $\boldsymbol{B}_\ell$

To commit to $\boldsymbol{x} \in \{0,1\}^\ell$, use trapdoor for $\boldsymbol{B}_\ell$ to sample $\boldsymbol{c}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell$ where

$$\boldsymbol{W}_1 \boldsymbol{c} = \boldsymbol{A}\boldsymbol{v}_1 + x_1 \boldsymbol{t}$$
$$\vdots$$
$$\boldsymbol{W}_\ell \boldsymbol{c} = \boldsymbol{A}\boldsymbol{v}_\ell + x_\ell \boldsymbol{t}$$

Verification relation

$$\sum_{i \in [\ell]} \alpha_i \boldsymbol{W}_i \boldsymbol{c} = \boldsymbol{A}\boldsymbol{v}_f + y \cdot \boldsymbol{t}$$

Opening to value $y = f(\boldsymbol{x}) = \sum_{i \in [\ell]} \alpha_i x_i$ is $\boldsymbol{v}_f \coloneqq \sum_{i \in [\ell]} \alpha_i \boldsymbol{v}_i$

Functional Commitments from Lattices

Security follows from ℓ-succinct SIS assumption [Wee24]:

SIS is hard with respect to $\boldsymbol{A}$ *given a trapdoor (a basis) for the matrix*

$$\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A} & & & \boldsymbol{W}_1 \\ & \ddots & & \vdots \\ & & \boldsymbol{A} & \boldsymbol{W}_\ell \end{array}\right]$$

where $\boldsymbol{A} \leftarrow \mathbb{Z}_q^{n \times m}$ and $\boldsymbol{W}_i \leftarrow \mathbb{Z}_q^{n \times m}$

Falsifiable assumption but does not appear to reduce to standard SIS

$\ell = 1$ case does follow from plain SIS (and when $\boldsymbol{W}_i$ is <u>very</u> wide)

Open problem: Understanding security or attacks when $\ell > 1$

Functional Commitments from Lattices

Security follows from ℓ-succinct SIS assumption [Wee24]:

SIS is hard with respect to $\boldsymbol{A}$ given a trapdoor (a basis) for the matrix

$$\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A} & & & \boldsymbol{W}_1 \\ & \ddots & & \vdots \\ & & \boldsymbol{A} & \boldsymbol{W}_\ell \end{array}\right]$$

where $\boldsymbol{A} \leftarrow \mathbb{Z}_q^{n \times m}$ and $\boldsymbol{W}_i \leftarrow \mathbb{Z}_q^{n \times m}$

Equivalent formulation:

SIS is hard with respect to $\boldsymbol{A}$ given $\boldsymbol{A}^{-1}(\boldsymbol{W}_i \boldsymbol{R})$ along with $\boldsymbol{W}_i, \boldsymbol{R}$

where $\boldsymbol{A} \leftarrow \mathbb{Z}_q^{n \times m}$, $\boldsymbol{W}_i \leftarrow \mathbb{Z}_q^{n \times m}$, and $\boldsymbol{R} \leftarrow D_{\mathbb{Z},s}^{m \times k}$ where $k \geq m(\ell + 1)$

Functional Commitments from Lattices

Linear functional commitments extends readily to support (bounded-depth) circuits

$$\boxed{\begin{matrix} \boldsymbol{W}_1 \boldsymbol{c} = \boldsymbol{A}\boldsymbol{v}_1 + x_1 \boldsymbol{t} \\ \vdots \\ \boldsymbol{W}_\ell \boldsymbol{c} = \boldsymbol{A}\boldsymbol{v}_\ell + x_\ell \boldsymbol{t} \end{matrix}} \quad \Longrightarrow \quad \boxed{\begin{matrix} \boldsymbol{W}_1 \boldsymbol{C} = \boldsymbol{A}\boldsymbol{V}_1 + x_1 \boldsymbol{G} \\ \vdots \\ \boldsymbol{W}_\ell \boldsymbol{C} = \boldsymbol{A}\boldsymbol{V}_\ell + x_\ell \boldsymbol{G} \end{matrix}}$$

Supports openings to linear functions

Supports openings to Boolean circuits

In this setting, $(\boldsymbol{W}_1 \boldsymbol{C}, \dots, \boldsymbol{W}_\ell \boldsymbol{C})$ is a [GVW14] homomorphic commitment to $\boldsymbol{x}$ (can be opened to any function $f(\boldsymbol{x})$ of bounded depth)

[see paper for details]

Can be sampled using **same** trapdoor for B_ℓ (security still reduces to ℓ-succinct SIS)

Summary of Functional Commitments

New methodology for constructing lattice-based commitments:

1. Write down the main verification relation ($\boldsymbol{c} = \boldsymbol{A}_i \boldsymbol{v}_i + x_i \boldsymbol{t}_i$)
2. **Publish** a trapdoor for the linear system induced by the verification relation

Security analysis relies on new q-type variants of SIS:

*SIS with respect to $\boldsymbol{A}$ is hard given a trapdoor for a **related** matrix $\boldsymbol{B}$*

"Random" variant of the assumption implies vector commitments and reduces to SIS

"Structured" variant (ℓ-succinct SIS) implies functional commitments for circuits

- Structure also enables **aggregating** openings

[see paper for details]

ℓ-Succinct SIS (and LWE)

SIS (or LWE) is hard with respect to $\boldsymbol{A}$ given a trapdoor (a basis) for the matrix

$$\boldsymbol{B}_\ell = \left[\begin{array}{ccc|c} \boldsymbol{A} & & & \boldsymbol{W}_1 \\ & \ddots & & \vdots \\ & & \boldsymbol{A} & \boldsymbol{W}_\ell \end{array}\right]$$

where $\boldsymbol{A} \leftarrow \mathbb{Z}_q^{n\times m}$ and $\boldsymbol{W}_i \leftarrow \mathbb{Z}_q^{n\times m}$

Falsifiable assumption that is implied by evasive LWE

Less structured assumption than k-R-ISIS or $\mathrm{BASIS}_{\mathrm{struct}}$ from recent works:

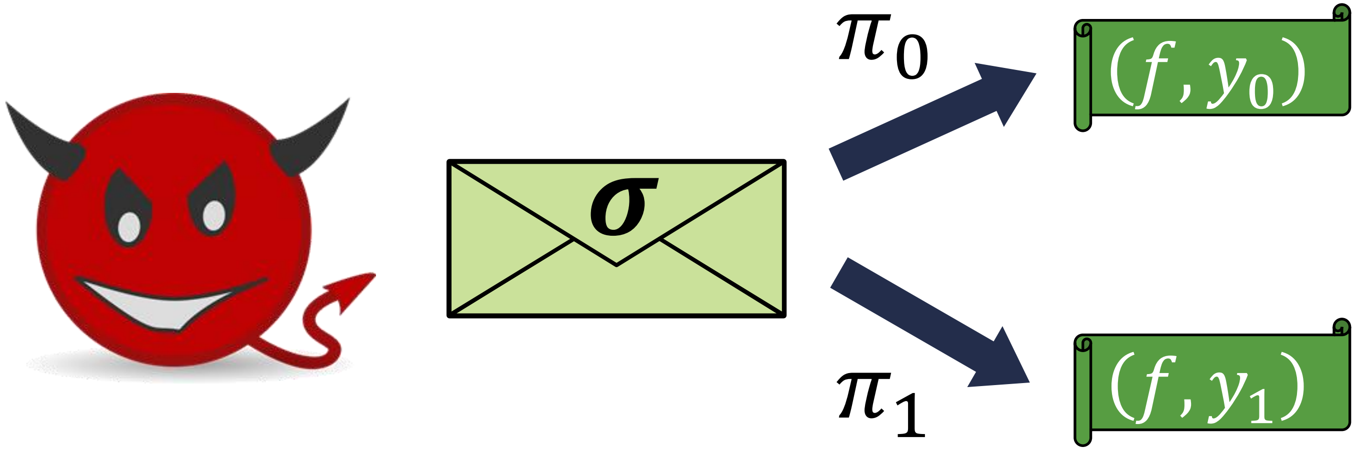
$$\boldsymbol{A}^{-1}(\boldsymbol{W}_i\boldsymbol{R}) \text{ where } \boldsymbol{W}_i \leftarrow \mathbb{Z}_q^{n\times m} \text{ and } \boldsymbol{R} \leftarrow D_{\mathbb{Z},s}^{m\times m(\ell+1)}$$

Can be used to get ABE with short ciphertexts (and broadcast encryption) [Wee24], functional commitments [WW23b], distributed broadcast encryption [CW24]

Cryptanalysis of Lattice-Based Knowledge Assumptions

Extractable Functional Commitments

Binding: efficient adversary cannot open σ to two different values with respect to the **same** f



$$\mathrm{Verify}(\mathrm{crs}, \sigma, (f, y_0), \pi_0) = 1$$

$$\mathrm{Verify}(\mathrm{crs}, \sigma, (f, y_1), \pi_1) = 1$$

Scheme could be binding, but still allow an efficient adversary to construct (malformed) commitment σ and opening to value 1 with respect to the **all-zeroes** function

Extractable Functional Commitments

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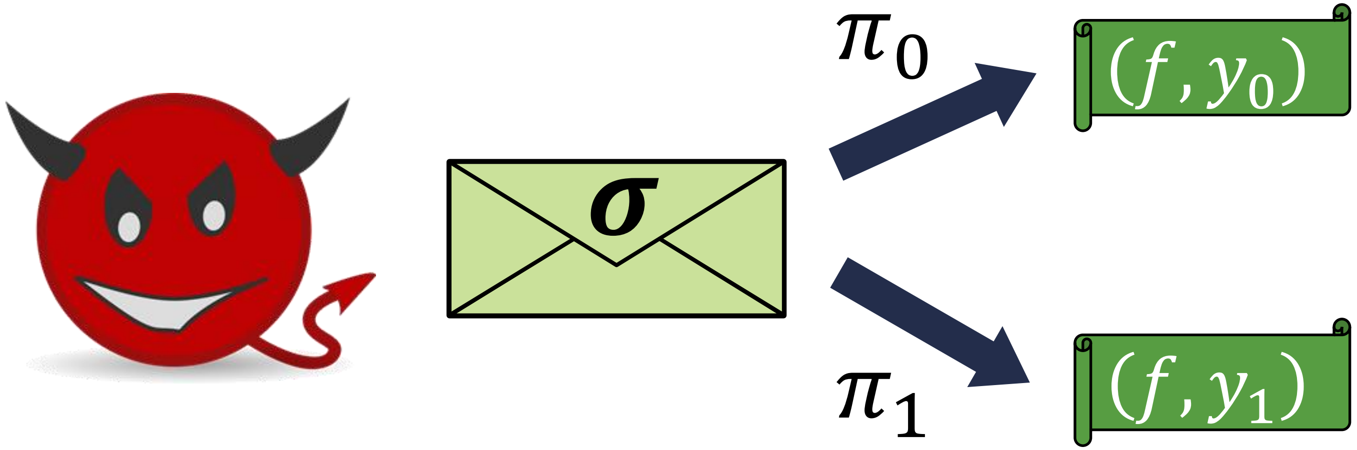

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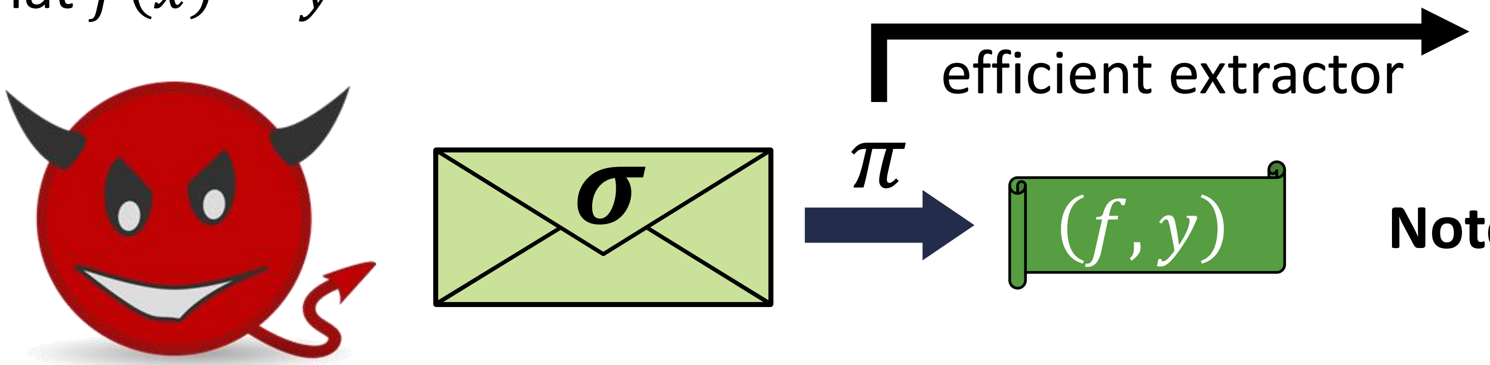
Extractability: efficient adversary that opens σ to y with respect to f must **know** an x such that $f(x) = y$



x such that $y = f(x)$

Note: f could have multiple outputs

Extractable Functional Commitments

Binding: efficient adversary cannot open σ to two different values with respect to the **same** f

Notion is equivalent to SNARKs, so will be challenging to build from a falsifiable assumption

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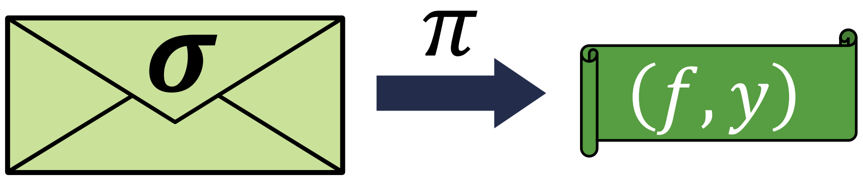

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Cryptanalysis of Lattice-Based Knowledge Assumptions

Typical lattice-based knowledge assumption (to get extractable commitments / SNARKs):

[ACLMT22]

given (tall) matrices $\boldsymbol{A}, \boldsymbol{D}$ and short preimages $\boldsymbol{Z}$ of a random target $\boldsymbol{T}$

if adversary can produce a short vector $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v}$ is in the image of $\boldsymbol{D}$ (i.e., $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{D}\boldsymbol{c}$), then there exists an extractor that outputs short $\boldsymbol{x}$ where $\boldsymbol{v} = \boldsymbol{Z}\boldsymbol{x}$

Observe: $\boldsymbol{A}\boldsymbol{v}$ for a random (short) $\boldsymbol{v}$ is outside the image of $\boldsymbol{D}$ (since $\boldsymbol{D}$ is tall)

Cryptanalysis of Lattice-Based Knowledge Assumptions

Typical lattice-based knowledge assumption (to get extractable commitments / SNARKs):

22]

For extractable functional commitments:

- $\boldsymbol{Z}$ is in the CRS
- Commitment is $\boldsymbol{c} = \boldsymbol{T}\boldsymbol{x}$
- Opening is $\boldsymbol{v}$ where $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{D}\boldsymbol{c}$

Extractable since valid opening can be associated with an honestly-generated commitment

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Obliviously Sampling a Solution

Typical lattice-based knowledge assumption (to get extractable commitments / SNARKs):

[ACLMT22]

Our work: algorithm to **obliviously** sample a solution $\boldsymbol{Av} = \boldsymbol{Dc}$ without knowledge of a linear combination $\boldsymbol{v} = \boldsymbol{Zx}$

Rewrite $\boldsymbol{AZ} = \boldsymbol{DT}$ as

$$[\boldsymbol{A} \mid \boldsymbol{DG}] \cdot \begin{bmatrix} \boldsymbol{Z} \\ -\boldsymbol{G}^{-1}(\boldsymbol{T}) \end{bmatrix} = \boldsymbol{0}$$

If $\boldsymbol{Z}$ and $\boldsymbol{T}$ are wide enough, we (heuristically) obtain a basis for $[\boldsymbol{A} \mid \boldsymbol{DG}]$

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Oblivious sampler (Babai rounding):

1. Take any (non-zero) integer solution $\boldsymbol{y}$ where $[\boldsymbol{A} \mid \boldsymbol{DG}]\boldsymbol{y} = \mathbf{0} \bmod q$
2. Assuming $\boldsymbol{B}^*$ is full-rank over $\mathbb{Q}$, find $\boldsymbol{z}$ such that $\boldsymbol{B}^*\boldsymbol{z} = \boldsymbol{y}$ (over $\mathbb{Q}$)
3. Set $\boldsymbol{y}^* = \boldsymbol{y} - \boldsymbol{B}^*\lfloor \boldsymbol{z} \rceil = \boldsymbol{B}^*(\boldsymbol{z} - \lfloor \boldsymbol{z} \rceil)$ and parse into $\boldsymbol{v}, \boldsymbol{c}$

Correctness: $[\boldsymbol{A} \mid \boldsymbol{DG}] \cdot \boldsymbol{y}^* = [\boldsymbol{A} \mid \boldsymbol{DG}] \cdot \boldsymbol{B}^*(\boldsymbol{z} - \lfloor \boldsymbol{z} \rceil) = \mathbf{0} \bmod q$ and $\boldsymbol{y}^*$ is short

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This solution is obtained by "rounding" off a <u>long</u> solution

Question: Can we explain such solutions as taking a <u>short</u> linear combination of $\boldsymbol{Z}$ (i.e., what the knowledge assumption asserts)

Correctness: $[\boldsymbol{A} \mid \boldsymbol{DG}] \cdot \boldsymbol{y}^* = [\boldsymbol{A} \mid \boldsymbol{DG}] \cdot \boldsymbol{B}^*(\boldsymbol{z} - \lfloor \boldsymbol{z} \rceil) = \boldsymbol{0} \bmod q$ and $\boldsymbol{y}^*$ is short

Template for Analyzing Lattice-Based Knowledge Assumptions

1. Start with the key verification relation (i.e., knowledge of a short solution to a linear system)
2. Express verification relation as finding non-zero vector in the kernel of a lattice defined by the verification equation
3. Use components in the CRS to derive a basis for the related lattice

1

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{D}\boldsymbol{c}$$

2

$$[\boldsymbol{A} \mid \boldsymbol{D}\boldsymbol{G}] \begin{bmatrix} \boldsymbol{v} \\ -\boldsymbol{G}^{-1}(\boldsymbol{c}) \end{bmatrix} = \boldsymbol{0}$$

3

$$[\boldsymbol{A} \mid \boldsymbol{D}\boldsymbol{G}] \cdot \begin{bmatrix} \boldsymbol{Z} \\ -\boldsymbol{G}^{-1}(\boldsymbol{T}) \end{bmatrix} = \boldsymbol{0}$$

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Implications:

- Oblivious sampler for integer variant of knowledge k-R-ISIS assumption from [ACLMT22]
  Implementation by Martin: `https://gist.github.com/malb/7c8b86520c675560be62eda98dab2a6f`
- Breaks extractability of the (integer variant of the) linear functional commitment from [ACLMT22] assuming hardness of inhomogeneous SIS (i.e., existence of efficient extractor for oblivious sampler implies algorithm for inhomogeneous SIS)

Open question: Can we extend the attacks to break soundness of the SNARK?

Template for Analyzing Lattice-Based Knowledge Assumptions

1. Start with the key verification relation (i.e., knowledge of a short solution to a linear system)
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Implications:

- Oblivious sampler for intege
  Implementation by Martin: `https`
- Breaks extractability of the (
  [ACLMT22] assuming hardn
  for oblivious sampler implies algorithm for inhomogeneous SIS)

The SNARK considers extractable commitment for **quadratic** functions while our current oblivious sampler only works for **linear** functions in the case of [ACLMT22]

Open question: Can we extend the attacks to break soundness of the SNARK?

Open Questions

Understanding the hardness of ℓ-succinct SIS/LWE (hardness reductions or cryptanalysis)?

Martin's blog post: `https://malb.io/sis-with-hints.html`

New applications of ℓ-succinct SIS/LWE?

Broadcast encryption, succinct ABE, succinct laconic function evaluation [Wee24]

Cryptanalysis of lattice-based SNARKs based on knowledge k-R-ISIS [ACLMT22, CLM23, FLV23]

Our oblivious sampler (heuristically) falsifies the assumption, but does not break existing constructions

Formulation of new lattice-based knowledge assumptions that avoids attacks

Thank you!

`https://eprint.iacr.org/2022/1515`

`https://eprint.iacr.org/2024/028`