The Master Theorem

Recurrences will be common in this course.

Multiplication: $T(n) = 4T(n/2) + n$

$T(n) = 3T(n/2) + n$

Sorting: $T(n) = 2T(n/2) + n$

Binary Search: $T(n) = T(n/2) + 1$

Simple Fibonacci: $T(n) = T(n-1) + T(n-2) + 1$

All with the
Base case: $T(n) = \Theta(1)$ for $n \leq O(1)$.

The master theorem: Formula to solve recurrence relations of the form

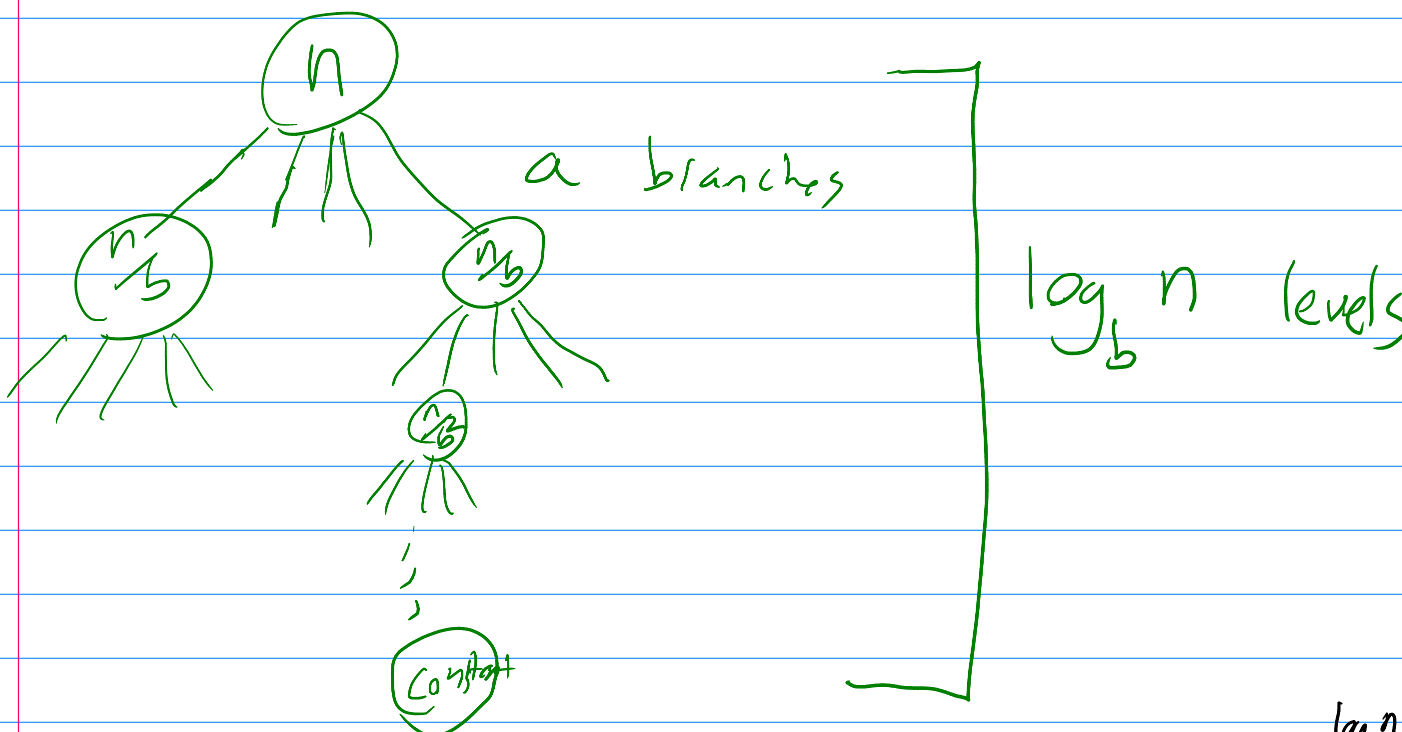
$$T(n) = a \cdot T(n/b) + f(n)$$

Simpler version: $f(n) = n.$

$$T(n) = n + a \cdot T(n/b)$$

$$= n + a\left[\frac{n}{b} + a \cdot T\left(\frac{n}{b^2}\right)\right]$$

$$= n + a\left[\frac{n}{b} + a\left(\frac{n}{b^2} + a \cdot T\left(\frac{n}{b^3}\right)\right)\right]$$



$$T(n) = n + \left(\frac{a}{b}\right)n + \left(\frac{a}{b}\right)^2 n + \left(\frac{a}{b}\right)^3 n + \ldots + \left(\frac{a}{b}\right)^{\log_b n} n$$

$$= \sum_{i=0}^{\log_b n} \left(\frac{a}{b}\right)^i \cdot n$$

Last term is $\left(\frac{a}{b}\right)^{\log_b n} \cdot n = a^{\log_b n} = n^{\log_b a}$

Simplify ✪ depending on a/b.

Case 1: $a < b$:

$$T(n) \leq \sum_{i=0}^{\infty} \left(\frac{a}{b}\right)^i n = \frac{n}{1-a/b} = O(n)$$

Case 2: $a = b$.

$$T(n) = \sum_{i=0}^{\log_b n} n = n \log_b n$$

Case 3: $a > b$

$$T(n) = \sum_{i=0}^{\log_b n} \left(n^{\log_b a}\right) \cdot \left(\frac{b}{a}\right)^i = O\left(n^{\log_b a}\right)$$

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what about

$$T(n) = aT(n/b) + n^2 \quad ?$$

$$= n^2 + a\left(\left(\frac{n}{b}\right)^2 + a\left(\left(\frac{n}{b^2}\right)^2 + a(\ldots)\right)\right)$$

Threshold between n^2 and $n^{\log_b a}$ is when $b^2 = a$.

$$n^2 = n^{\log_b a}$$

Master Theorem

$$T(n) = aT(n/b) + f(n), \qquad a, b \text{ constants}$$

Define $c_{crit} = \log_b a$.

Case 1: $f(n) = O(n^c)$ for $c < c_{crit}$

Then $T(n) = \Theta(n^{c_{crit}})$

Case 2: $f(n) = \Theta(n^{c_{crit}} \cdot \log^k n)$, k constant

Then $T(n) = \Theta(n^{c_{crit}} \log^{k+1} n)$

Case 3: $f(n) = \Omega(n^c)$ for $c > c_{crit}$

Then $T(n) = \Theta(f(n))$ (*)

(*) assuming "regularity condition" $af(\frac{n}{b}) < (1-\varepsilon) f(n)$ for some $\varepsilon > 0$, $\forall$ sufficiently large n.
In practice this will always hold.