Dijkstra Variants: A* and Potentials

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UT Austin

CS 331, Spring 2020 Coronavirus Edition

Class Outline

1. Bottleneck Shortest Paths

2. A* search

3. Problems

Logistics

- “Raise hand” will hopefully not crash my connection now.

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- We're going to try using Zoom breakout rooms for problems, later today.
  - Inside, you can "Ask for help" and it pops up a notification for me.
  - You stop being able to see my screen, so be sure to record the exercises before joining the breakout room.

Talk Outline

1. Bottleneck Shortest Paths

2. A* search

3. Problems

Bottleneck Shortest Paths

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- On undirected graph: is maximum spanning tree
- On directed graph: Dijkstra/Prim variant solves in $O(E + V \log V)$.

Dijkstra's Algorithm

```
function DIJKSTRA(s)
    pred, dist ← {}, {}
    q ← PRIORITYQUEUE([(0, s, None)])          ▷ dist, vertex, pred
    while q do
        d, u, parent ← q.pop_min()
        if u ∈ pred then
            continue
        pred[u] ← parent
        dist[u] ← d
        for u → v ∈ E do
            q.push( (dist[u] + w(u → v), v, u) )
    return dist, pred
```

~~Dijkstra's~~ Prim's Algorithm

```
function PRIM(s)
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        d, u, parent ← q.pop_max()
        if u ∈ pred then
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        pred[u] ← parent
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        for u → v ∈ E do
            q.push( (min(dist[u], w(u → v)), v, u) )
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$$(\min, +) \to (\max, \min)$$

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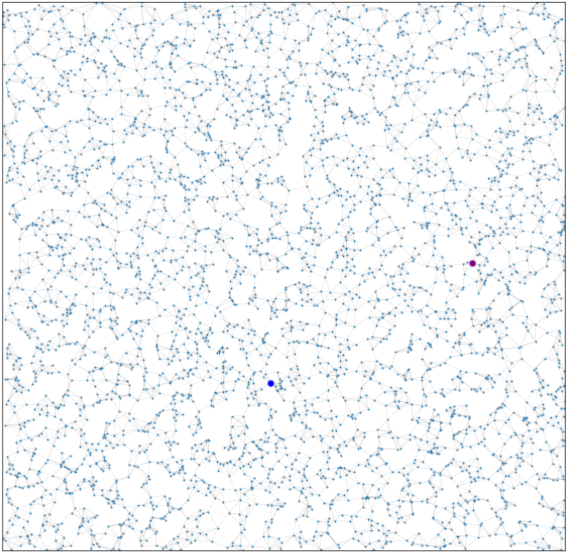
Shortest $s - t$ path with Dijkstra



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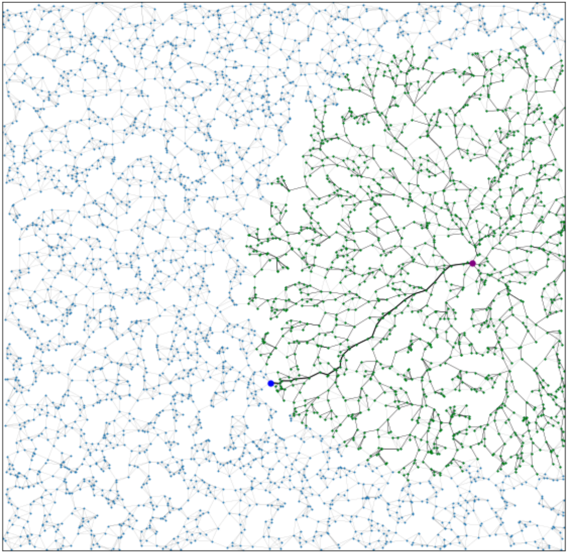

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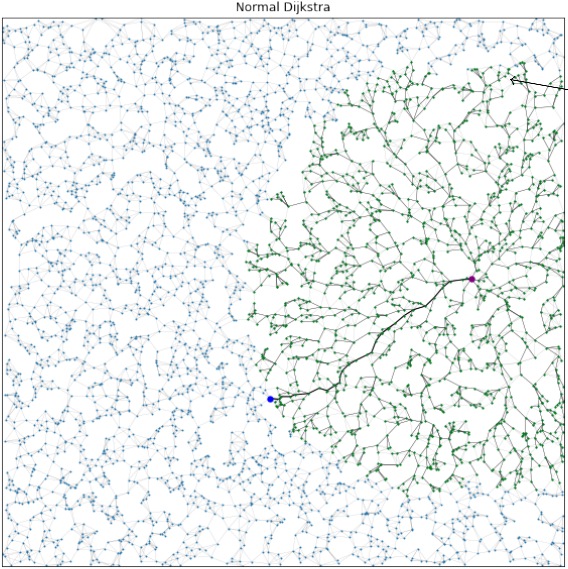

This is the wrong direction. Why waste our timing exploring it?

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- Example: $h(\text{NYC})$ is Euclidean distance from NYC to SF.
  - Any path through NYC will take *at least* 3700 miles.

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        d, u, parent ← q.pop_min()
        if d ≥ dist[u] then
            continue
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        for u → v ∈ E do
            q.push( (dist[u] + w(u → v), v, u) )
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function A*(s)
    pred, dist ← {}, {}
    q ← PriorityQueue([(0 + h(s), s, None)])    ▷ dist, vertex, pred
    while q do
        d, u, parent ← q.pop_min()
        if d−h(u) ≥ dist[u] then
            continue
        pred[u] ← parent
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        for u → v ∈ E do
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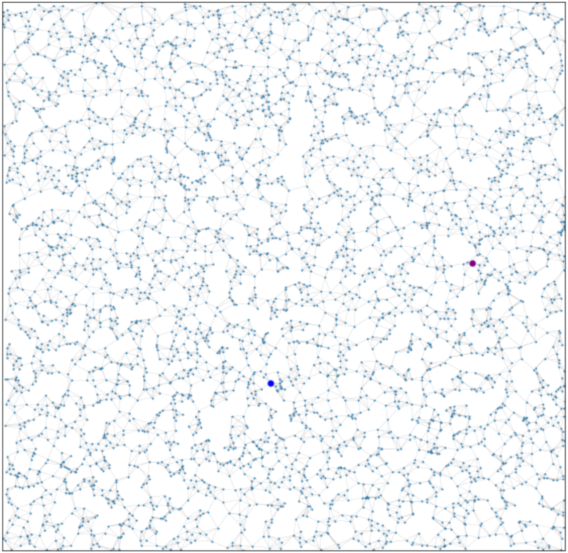

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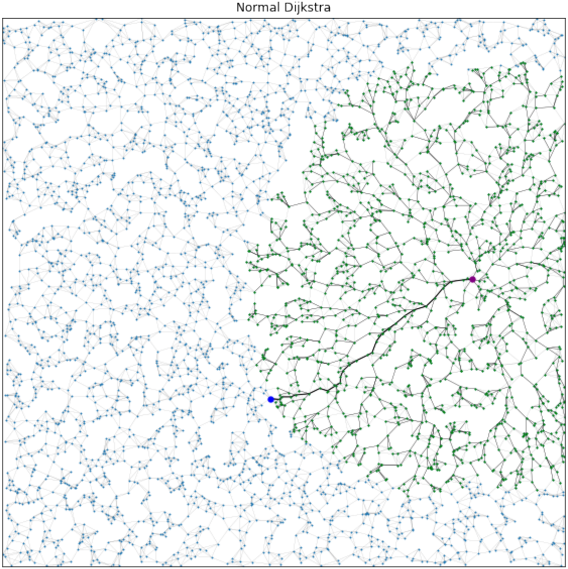

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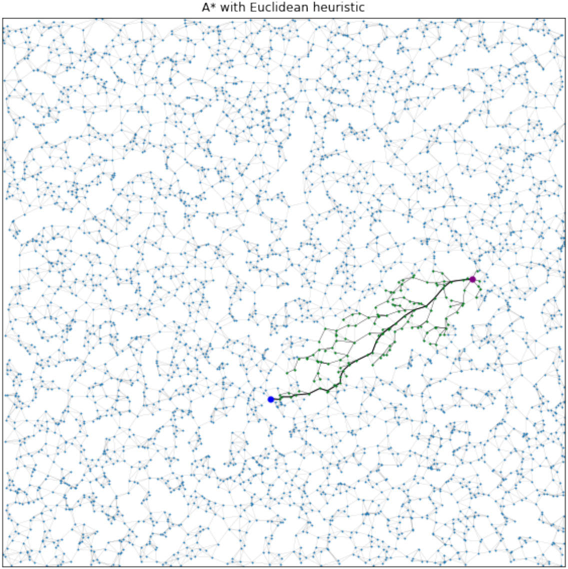

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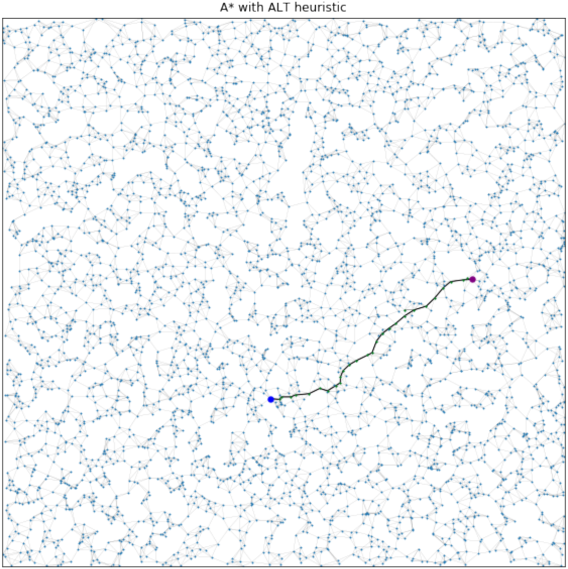

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  - Can be faster in many cases.

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Shortest Path Problems

http://jeffe.cs.illinois.edu/teaching/algorithms/book/08-sssp.pdf

- Problem 2: Dijkstra with k negative edges.
- Problem 3: *vertices*, not edges, have weight.
- Problem 5: edge reinsertion
- Problem 4: Replacement paths on directed graphs
- Problem 12: Smallest shortest path
- Problem 16, 17: Remember reductions?
- Problem 1 of https://www.cs.utexas.edu/~ecprice/courses/331h/psets/331h-ps6.pdf