Problem Set 4

Randomized Algorithms

Due Wednesday, November 4

1. [MR 4.9]. Consider the following randomized variant of the bit fixing algorithm. Each packet randomly orders the bit positions in the label of its source and then corrects the mismatched bits in that order. Show that there is a permutation for which, with high probability, this algorithm uses $2^{\Omega(n)}$ steps to route.

2. [MR7.2]. Two rooted trees T_1 and T_2 are said to be isomorphic if there exists a one to one mapping f from the nodes of T_1 to those of T_2 satisfying the following condition: v is a child of w in T_1 if and only if $f(v)$ is a child of $f(w)$ in T_2. Observe that no ordering is assumed on the children of any vertex. Devise an efficient randomized algorithm for testing the isomorphism of rooted trees and analyze its performance. **Hint**: Recursively associate a polynomial P_v with each vertex v in a tree T.

3. [Karger] Suppose you are given a graph whose edge lengths are all integers in the range from 0 to B. Suppose also that you are given the all-pairs distance matrix for this graph (it can be constructed by a variant of Seidel's deterministic distance algorithm). Prove that you can identify the (successor matrix representation of the) shortest paths in $O(B^2 MM(n) \log^2 n)$ time, where $MM(n)$ is the time to multiply $n \times n$ matrices.

4. Let S be an unknown set of n items (with n known). Suppose that you receive a sample T of k items chosen from S uniformly at random *without replacement*. Show how to construct a sample T' of k items from S, whose distribution is identical to a uniform sample of k items from S drawn *with replacement*.

5. [Moshkovitz] Suppose that you have a giant (i.e. infinite) bag of coins. You know that 90% of the coins are highly biased, and come up heads 90% of the time. The other 10% of coins are unbiased, and come up heads 50% of the time. You do not know which coins are which, and you would like to find one of the biased coins.

   You are allowed to flip coins n times – each coin you flip can be either a fresh random coin from the bag, or a coin that you have flipped before. At the end of n coin flips, you must output a coin. You succeed if the coin is biased, and fail if the coin is unbiased. What is the minimum probability of failure, and how can you achieve this?

   (a) Show that the failure probability must be at least $\exp(-O(n))$.

   (b) Suppose that the biased coins were actually 100% biased. Show how to achieve $\exp(-\Omega(n))$ failure probability.

   (c) Show how to achieve $\exp(-\Omega(n))$ failure probability in the setting described, where the biased coins are 90% biased.

   (d) [Optional; this is an open research question.] Suppose that of the other 10% of coins, some may have arbitrary bias. You still only fail if the coin you output is completely unbiased. Show how to achieve $\exp(-\Omega(n))$ failure probability, or show that this is impossible.