CS 388R: Randomized Algorithms, Fall 2019 October 17th

Lecture 15: K-Hamiltonian Path; Sampling; median finding;

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NOTE: THESE NOTES HAVE NOT BEEN EDITED OR CHECKED FOR CORRECTNESS

1 K-Hamiltonian Path

Question: Randomized algorithm for finding a Hamiltonian path of length k in a given graph G.

1. Randomly k-color the graph.
2. Run deterministic algorithm to find the shortest path that visits k distinct colors. Using dynamic programming. [STATE = where you end up & Which colors have seen so far.] Can be done in $n^2 \cdot 2^k$ time, for n steps and 2^k states.
3. Repeat $\log(\frac{1}{\delta})e^k$ times.

Analysis: We only care about coloring true path/set of k nodes. The chance of having a correct Hamiltonian path of length k(correct coloring) is

$$\frac{\text{\# of valid coloring of the set}}{\text{\# of total coloring}} = \frac{k!}{k^k} \approx \frac{1}{e^k}$$

If we repeat $\log(\frac{1}{\delta})e^k$ times, we'll get the correct result with high probability $(1-\delta)$ The total time taken is:

$$O(n^2 \cdot 2^{O(k)})$$

2 Sampling

Question: There is some $S \subseteq SPACE(U)$. We have an oracle to query if $x \in S$ for $\forall x$. Goal: estimate $Vol(S)$.

Simple algorithm: Pick $x_1, x_2, \cdots, x_m \in U$ uniformly, and query if $x_i \in S$. Let Z_i be the indicator event whether $x_i \in S$. Then,

$$\frac{\text{\# lie in S}}{\text{\# picked}} \approx \frac{\text{Vol(S)}}{\text{Vol(U)}} = p$$

There are many factors that p can depend upon. For example, it'll depend on how large S and U are. One could imagine the above process of sampling and estimating in 2-dimension. In high dimensional space, it'll look as estimating the volume of some $d-$dimensional polytope.

Question How many samples are needed to learn p with estimator satisfying $\tilde{p} \in (1 \pm \epsilon)p$ with probability $1 - \delta$. That is, an (ϵ, δ) approximation.

One could recollect that we're dealing with a similar event we have studied before - of tossing a coin and estimating probability of getting a head. Just for the sake of completeness, we'll derive the result here again. Let's assume we sample n points $x_1, x_2, \cdots, x_n$. Then, we know that the expected number of points lying in S will be np. That is,

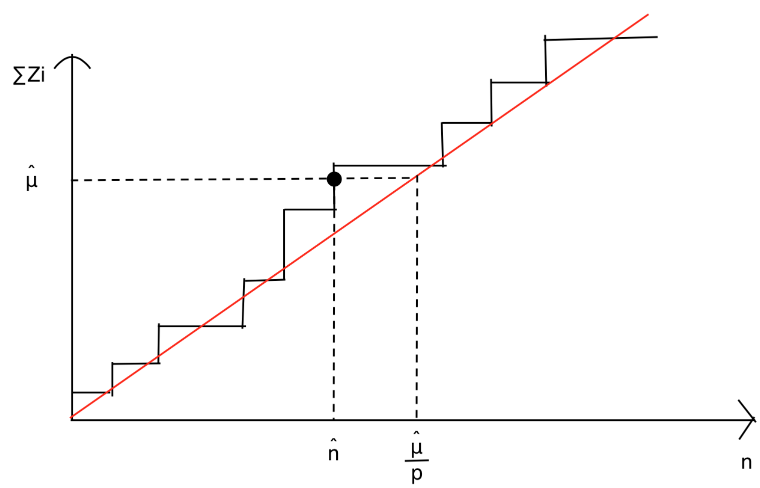
$$\mathbb{E}[\frac{\sum Z_i}{n}] = p$$

Then,

$$\begin{aligned}\mathbb{P}\left[\left|\frac{\sum_{i=1}^{n} Z_i}{n} - p\right| > p\epsilon\right] &= \mathbb{P}[|\sum_{i=1}^{n} Z_i - np| > np\epsilon] \\ &\leq 2e^{-\frac{\epsilon^2}{3}np}\end{aligned}$$

Thus, in order for this probability to be less than δ, we get $n \geq \frac{3}{p\epsilon^2}\log(\frac{2}{\delta})$.

One might be tempted to sample $\hat{n}$ elements such that $\sum_{i=1}^{\hat{n}} Z_i = \frac{3}{p\epsilon^2}\log(\frac{2}{\delta})$, and estimate the probability p as $\tilde{p} = \frac{\sum_{i=1}^{\hat{n}} Z_i}{\hat{n}}$. But we can't be sure that this is indeed a correct estimation. Consider the following picture.



Where $\hat{\mu} = \frac{3}{\epsilon^2}\log(\frac{2}{\delta})$, and the red line represents the actual $\sum Z_i$ vs n curve. For $\sum Z_i = \hat{\mu}$, the actual n value is $n = \hat{\mu}/p$, whereas we get the number of samples where $\sum Z_i = \hat{\mu}$ as $\hat{n}$. Hence, we estimate

$$\begin{aligned}\tilde{p} &= \frac{\hat{\mu}}{\hat{n}} \\ \implies \hat{n} &= \frac{\hat{\mu}}{\tilde{p}} \\ \implies \hat{n} &\in \frac{\hat{\mu}}{p}\left[\frac{1}{1+\epsilon}, \frac{1}{1-\epsilon}\right]\end{aligned}$$

The previous result tell us about the accuracy of $\sum Z_i$, that is, the value of $\sum Z_i$ will be within $(1 \pm \epsilon)$ actual mean $\hat{\mu}$ (w.h.p.). What we moreover need is that the number of samples $\hat{n}$ is within the range as specified above. We'll prove that it's indeed in this range with high probability.

Consider the number of samples $n' = \frac{\hat{\mu}}{p(1-\epsilon)}$. For this, the actual mean is $\mu = \frac{\hat{\mu}}{(1-\epsilon)}$. We need to show that $\mathbb{P}\left[\sum_{i=1}^{n'} Z_i < \hat{\mu}\right] < \delta$.

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{n'} Z_i < \frac{\hat{\mu}}{1-\epsilon}(1-\epsilon)\right] &\leq e^{-\frac{\epsilon^2}{3}n'p} \\ &= e^{-\frac{1}{(1-\epsilon)}log(\frac{2}{\delta})} \\ &\leq \delta\end{aligned}$$

Thus, with very high probability $\hat{n}$ will be less than $n' = \frac{\hat{\mu}}{p(1-\epsilon)}$. Similarly, we can prove for $\frac{\hat{\mu}}{p(1+\epsilon)}$ and hence with high probability $\hat{n} \in \frac{\hat{\mu}}{p}\left[\frac{1}{1+\epsilon}, \frac{1}{1-\epsilon}\right]$. Thus, we guarantee that sampling $\hat{n}$ elements such that $\sum_{i=1}^{\hat{n}} Z_i = \frac{3}{\epsilon^2}\log(\frac{2}{\delta})$ elements gives a probability estimation $\tilde{p} = \frac{\sum_{i=1}^{\hat{n}} Z_i}{\hat{n}}$ satisfying $\tilde{p} \in p(1 \pm \epsilon)$ with high probability.

3 Median Finding

Question: Given $x_1, ..., x_n$, find the median x_i.

1. Sort & output median $\rightarrow O(n \log n)$.
2. Quick select, modified quick sort $T(n) = O(n) + T(\frac{3n}{4}) \rightarrow O(n)$ time and # of comparisons in expectation. Still has $(\frac{1}{k})^k$ chance of $\Theta(nk)$ work.
3. Fancy deterministic algorithm: split $\frac{n}{5}$ sets of 5 elements each, apply the same divide and conquer method. Take the median of medians.

$$T(n) = O(n) + T(\frac{n}{5}) + T(\frac{7n}{10}) \rightarrow O(n)$$

Randomized Algorithm in $O(n)$ w.h.p.:

Sample $y_1, y_2, ..., y_s$ from $X[y_i = x_j$ for $j \in [n]$ uniformly at random]. Sort in $O(s \log s)$. We want to say

$$y_{\frac{s}{2}-k} \leq \text{median x} \leq y_{\frac{s}{2}+k}$$

w.h.p for $k = \boldsymbol{O(\sqrt{S \log n})}$.
Pr[$y_{\frac{s}{2}-k} >$ median x]=Pr[at least $\frac{s}{2} + k$ elements choices of y $\leq$ median x]
Using indicator $Z_i = (y_i \leq \text{median X})$

$$Pr[Z_i] = \frac{1}{2}$$
$$E[\sum Z_i] = \frac{s}{2}$$

$$Pr[\sum Z_i \geq \frac{s}{2} + k] \leq e^{\frac{-2k^2}{s}}$$

Using the value $k = O(\sqrt{S \log n})$, we get the above probability being very low.

Question: How do we use this algorithm?

Option1: Use $y_{\frac{s}{2}}$ for quick select. Rank of $y_{\frac{s}{2}}$ is $\frac{n}{2} \pm O(n\sqrt{\frac{\log n}{2}})$ w.h.p.

Option2: Scan through x, and put them in one of the following groups: $(x < y_L)$, $(x \in [y_L, y_H])$, or $(x > y_H)$) for $(L, H) = (\frac{s}{2} - k, \frac{s}{2} + k)$. Sort $x \in [y_L, y_H]$ and output the $(\frac{y}{2} - |[x < y_L]|)^{th}$ element.

$$\begin{aligned}
\#\text{ of comparisons } &\leq O(s \log s) \leftarrow \text{sort y} \\
&+ \leq 2n \leftarrow \text{partition} \\
&+ O(m \log m)
\end{aligned}$$

where $m = |(x|x \in [y_L, y_H])|$. Notice that the $2n$ term is actually $1.5n$ as for almost half of the elements, we only compare with y_L.
Consider the following equation, which holds true for any fraction $f \in [0, 1]$

$$y_{f \cdot s - k} \leq x_{(f \cdot n)} \leq y_{f \cdot s + k} \quad \forall \text{fractions } f$$

What we want are the number of such x such that

$$(x|x \in [y_{\frac{s}{2} - k}, y_{\frac{s}{2} + k}])$$

This only happens for x_{fn} with $f \cdot s \geq \frac{1}{2}s - 2k$

$$\implies f = \frac{1}{2} - \frac{2k}{s} \text{ to } \frac{1}{2} + \frac{2k}{s}$$

Therefore, $(m\log(m)) \leq \frac{4k}{s} \cdot n = O(4n\frac{\sqrt{\log n}}{s}) = O(n\frac{\sqrt{\log n}}{s})$

Pick $\log n << s << \frac{n}{\log n}$. $\implies$ # of comparisons is $1.5n + O(n)$
$s = n^{\frac{2}{3}} \implies 1.5n + O(n^{\frac{2}{3}\log n})$