Problem Set 3

Sublinear Algorithms

Due Thursday, October 20

1. Recall that $M(X, d, \epsilon)$ denotes the packing number for space X with distance d and radius ϵ, and $N(X, d, \epsilon)$ denotes the covering number. Prove that
$$M(X, d, 2\epsilon) \leq N(X, d, \epsilon) \leq M(X, d, \epsilon)$$

2. In this problem we show that matrices that satisfy the RIP-2 cannot be very sparse. Let $A \in \mathbb{R}^{m \times n}$ satisfy the $(k, 1/2)$ RIP for $m < n$. Suppose that the average column sparsity of A is d, i.e. A has nd nonzero entries.

   Furthermore, suppose that $A \in \{0, \pm\alpha\}^{m \times n}$ for some parameter α.

   (a) By looking at the sparsest column, give a bound for α in terms of d.

   (b) By looking at the densest row, give a bound for α in terms of n, m, d and k.

   (c) Conclude that either $d \gtrsim k$ or $m \gtrsim n$. (Recall that this means: there exists a constant C for which $d \geq k/C$.)

   (d) What if each non-zero $A_{i,j}$ were drawn from $N(0, 1)$?

   (e) [Optional] Extend the result to general settings of the non-zero $A_{i,j}$.

3. In class we have shown various algorithms for sparse recovery that tolerate noise and use $O(k \log(n/k))$ measurements, and shown that any ℓ_1/ℓ_1 sparse recovery algorithm must use this many measurements. But what if we don't care about tolerating noise, and only want to recover x from Ax when x is exactly k-sparse?

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{2k-1} & \alpha_2^{2k-1} & \dots & \alpha_n^{2k-1} \end{pmatrix}$$

for distinct α_i.

(a) Prove that any $2k \times 2k$ submatrix of A is invertible.

(b) Give an $n^{O(k)}$ time algorithm to recover x from Ax under the assumption that x is k-sparse.

(c) [Optional] Give an $n^{O(1)}$ time algorithm to recover x from Ax under the assumption that x is k-sparse. You may choose specific values for the α_i. Hint: look up syndrome decoding of Reed-Solomon codes.