Problem Set 8

Sublinear Algorithms

Due Thursday, November 12

1. In class we have shown various algorithms for sparse recovery that tolerate noise and use $O(k \log(n/k))$ measurements, and shown that any ℓ_1/ℓ_1 sparse recovery algorithm must use this many measurements. But what if we don't care about tolerating noise, and only want to recover x from Ax when x is exactly k-sparse?

   Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{2k-1} & \alpha_2^{2k-1} & \dots & \alpha_n^{2k-1} \end{pmatrix}$$

   for distinct α_i.

   (a) Prove that any $2k \times 2k$ submatrix of A is invertible. (Hint: look up the Vandermonde determinant.)

   (b) Give an $n^{O(k)}$ time algorithm to recover x from Ax under the assumption that x is k-sparse.

   (c) [Optional] Give an $n^{O(1)}$ time algorithm to recover x from Ax under the assumption that x is k-sparse. You may choose specific values for the α_i. Hint: look up syndrome decoding of Reed-Solomon codes.

2. In order to show that SSMP makes progress in each stage, we used a lemma that we will show in this problem.

Let $x_1, \ldots, x_k \in \mathbb{R}^d$, and suppose that

$$\sum_{i=1}^{k} \|x_i\|_1 \leq (1+\delta) \| \sum_{i=1}^{k} x_i \|_1$$

for some small enough δ (say, $\delta = 1/10$). In some sense, this is saying that there is not much "slack" in they are lined up head-to-tail.

(a) Let $z = \sum_{i=1}^{k} x_i$. Show that $\mathrm{E}_{i \in [k]} \|z - x_i\|_1 \leq (1 - \frac{\Omega(1)}{k}) \|z\|_1$.

(b) Now suppose $z = \sum_{i=1}^{k} x_i + w$ for some $w \in \mathbb{R}^d$ with $\|w\|_1 \leq \epsilon \|z\|_1$ for small enough constant ϵ. Again, show that there exists an i such that $\|z - x_i\|_1 \leq (1 - \frac{\Omega(1)}{k}) \|z\|_1$.