Show that

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max(|\delta_0|, |\delta_1|).$$

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2$$

$$\left\|\begin{pmatrix}\delta_0 & 0\\ 0 & \delta_1\end{pmatrix}\right\|_2^2 \quad = \quad \max_{\|x\|_2=1}\left\|\begin{pmatrix}\delta_0 & 0\\ 0 & \delta_1\end{pmatrix}\begin{pmatrix}\chi_0\\ \chi_1\end{pmatrix}\right\|_2^2$$

$$
\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \end{pmatrix} \right\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left[|\delta_0 \chi_0|^2 + |\delta_1 \chi_1|^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left[|\delta_0 \chi_0|^2 + |\delta_1 \chi_1|^2 \right] \\
&= \max_{\|x\|_2=1} \left[|\delta_0|^2 |\chi_0|^2 + |\delta_1|^2 |\chi_1|^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left[|\delta_0 \chi_0|^2 + |\delta_1 \chi_1|^2 \right] \\
&= \max_{\|x\|_2=1} \left[|\delta_0|^2 |\chi_0|^2 + |\delta_1|^2 |\chi_1|^2 \right] \\
&\leq \max_{\|x\|_2=1} \left[\max(|\delta_0|, |\delta_1|)^2 \left[|\chi_0|^2 + |\chi_1|^2 \right] \right]
\end{aligned}
$$

$$\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left[|\delta_0 \chi_0|^2 + |\delta_1 \chi_1|^2 \right] \\
&= \max_{\|x\|_2=1} \left[|\delta_0|^2 |\chi_0|^2 + |\delta_1|^2 |\chi_1|^2 \right] \\
&\leq \max_{\|x\|_2=1} \left[\max(|\delta_0|, |\delta_1|)^2 \left[|\chi_0|^2 + |\chi_1|^2 \right] \right] \\
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\end{aligned}$$

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&\leq \max_{\|x\|_2=1} \left[\max(|\delta_0|, |\delta_1|)^2 \left[|\chi_0|^2 + |\chi_1|^2 \right] \right] \\
&= \max(|\delta_0|, |\delta_1|)^2
\end{aligned}
$$

Conclude: $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \leq \max(|\delta_0|, |\delta_1|)$.

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2$$

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2$$

$$
\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2 \\
&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2
\end{aligned}
$$

$$
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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 \\
&= \left\| \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\|_2
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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 \\
&= \left\| \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\|_2 \\
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\end{aligned}$$

$$
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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 \\
&= \left\| \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\|_2 \\
&= \left\| \delta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 \\
&= |\delta_1| \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2
\end{aligned}
$$

$$
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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 \\
&= \left\| \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\|_2 \\
&= \left\| \delta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 \\
&= |\delta_1| \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 = |\delta_1|
\end{aligned}
$$

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&= |\delta_1| \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 = |\delta_1|
\end{aligned}
$$

Conclude: $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \geq \max(|\delta_0|, |\delta_1|).$

More precisely:

Choose $j \in \{0, 1\}$ such that $|\delta_j| = \max(|\delta_0|, |\delta_1|)$.

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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2
\end{aligned}
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\begin{aligned}
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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2 \\
&= \|\delta_j e_j\|_2 \\
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&\geq \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2 \\
&= \|\delta_j e_j\|_2 \\
&= |\delta_j| \, \|e_j\|_2 \\
&= |\delta_j|
\end{aligned}
$$

More precisely:

Choose $j \in \{0,1\}$ such that $|\delta_j| = \max(|\delta_0|, |\delta_1|)$.

$$
\begin{array}{rcl}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 & = & \max\limits_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2 \\
& \geq & \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2 \\
& = & \|\delta_j e_j\|_2 \\
& = & |\delta_j| \, \|e_j\|_2 \\
& = & |\delta_j| \\
& = & \max(|\delta_0|, |\delta_1|).
\end{array}
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& \geq & \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2 \\
& = & \|\delta_j e_j\|_2 \\
& = & |\delta_j| \, \|e_j\|_2 \\
& = & |\delta_j| \\
& = & \max(|\delta_0|, |\delta_1|).
\end{array}
$$

Conclude: $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \geq \max(|\delta_0|, |\delta_1|)$.

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \leq \max(|\delta_0|, |\delta_1|)$$

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \leq \max(|\delta_0|, |\delta_1|)$$

and

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \geq \max(|\delta_0|, |\delta_1|).$$

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \leq \max(|\delta_0|, |\delta_1|)$$

and

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \geq \max(|\delta_0|, |\delta_1|).$$

Hence

$$\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max(|\delta_0|, |\delta_1|).$$

Key insight

Given: $f : D \to \mathbb{R}$.

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If we want to show that

$$\max_{x \in D} f(x) = \alpha$$

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Given: $f : D \to \mathbb{R}$.

If we want to show that

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then a strategy is

- ⋆ Show that
$$\max_{x \in D} f(x) \leq \alpha.$$
- ⋆ Find a specific $y \in D$ such
$$f(y) = \alpha.$$

Show that $\max_{x \in D} f(x) \leq \alpha$:

$$\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2^2 &= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left\| \begin{pmatrix} \delta_0 \chi_0 \\ \delta_1 \chi_1 \end{pmatrix} \right\|_2^2 \\
&= \max_{\|x\|_2=1} \left[|\delta_0 \chi_0|^2 + |\delta_1 \chi_1|^2 \right] \\
&= \max_{\|x\|_2=1} \left[|\delta_0|^2 |\chi_0|^2 + |\delta_1|^2 |\chi_1|^2 \right] \\
&\leq \max_{\|x\|_2=1} \left[\max(|\delta_0|, |\delta_1|)^2 \left[|\chi_0|^2 + |\chi_1|^2 \right] \right] \\
&= \max(|\delta_0|, |\delta_1|)^2
\end{aligned}$$

Conclude: $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 \leq \max(|\delta_0|, |\delta_1|).$

Find y such that $f(y) = \alpha$:

Choose $j \in \{0, 1\}$ such that $|\delta_j| = \max(|\delta_0|, |\delta_1|)$.

$$\begin{aligned}
\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} e_j \right\|_2 &= \|\delta_j e_j\|_2 \\
&= |\delta_j| \, \|e_j\|_2 \\
&= |\delta_j| = \max(|\delta_0|, |\delta_1|).
\end{aligned}$$

Conclude: $\left\| \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix} \right\|_2 = \max(|\delta_0|, |\delta_1|)$.