CGS

First iteration

$$\rho_{0,0} = \|a_0\|_2 = \sqrt{1+\varepsilon^2} = \sqrt{1+\varepsilon_{\text{mach}}}$$

which is rounded to 1**.**

$$q_0 = a_0/\rho_{0,0} = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{pmatrix} /1 = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{pmatrix}$$

Second iteration

$$\rho_{0,1} = q_0^H a_1 = 1$$

$$a_1^\perp = a_1 - \rho_{0,1} q_0 = \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}$$

$$\rho_{1,1} = \|a_1^\perp\|_2 = \sqrt{2\varepsilon^2} = \sqrt{2}\varepsilon$$

$$q_1 = a_1^\perp/\rho_{1,1} = \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{pmatrix} /(\sqrt{2}\varepsilon) = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

Third iteration

$$\rho_{0,2} = q_0^H a_2 = 1$$

$$\rho_{1,2} = q_1^H a_2 = 0$$

$$a_2^\perp = a_2 - \rho_{0,2} q_0 - \rho_{1,2} q_1 = \begin{pmatrix} 0 \\ -\varepsilon \\ 0 \\ \varepsilon \end{pmatrix}$$

$$\rho_{2,2} = \|a_2^\perp\|_2 = \sqrt{2\varepsilon^2} = \sqrt{2}\varepsilon$$

$$q_2 = a_2^\perp/\rho_{2,2} = \begin{pmatrix} 0 \\ -\varepsilon \\ 0 \\ \varepsilon \end{pmatrix} /(\sqrt{2}\varepsilon) = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

MGS

First iteration

$$\rho_{0,0} = \|a_0\|_2 = \sqrt{1+\varepsilon^2} = \sqrt{1+\varepsilon_{\text{mach}}}$$

which is rounded to 1**.**

$$q_0 = a_0/\rho_{0,0} = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{pmatrix} /1 = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{pmatrix}$$

Second iteration

$$\rho_{0,1} = q_0^H a_1 = 1$$

$$a_1^\perp = a_1 - \rho_{0,1} q_0 = \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}$$

$$\rho_{1,1} = \|a_1^\perp\|_2 = \sqrt{2\varepsilon^2} = \sqrt{2}\varepsilon$$

$$q_1 = a_1^\perp/\rho_{1,1} = \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{pmatrix} /(\sqrt{2}\varepsilon) = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

Third iteration

$$\rho_{0,2} = q_0^H a_2 = 1$$

$$a_2^\perp = a_2 - \rho_{0,2} q_0 = \begin{pmatrix} 0 \\ -\varepsilon \\ 0 \\ \varepsilon \end{pmatrix}$$

$$\rho_{1,2} = q_1^H a_2^\perp = (\sqrt{2}/2)\varepsilon$$

$$a_2^\perp = a_2^\perp - \rho_{1,2} q_1 = \begin{pmatrix} 0 \\ -\varepsilon/2 \\ -\varepsilon/2 \\ \varepsilon \end{pmatrix}$$

$$\rho_{2,2} = \|a_2^\perp\|_2 = \sqrt{(6/4)\varepsilon^2} = (\sqrt{6}/2)\varepsilon$$

$$q_2 = a_2^\perp/\rho_{2,2} = \begin{pmatrix} 0 \\ -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} \\ \varepsilon \end{pmatrix} /(\frac{\sqrt{6}}{2}\varepsilon) = \begin{pmatrix} 0 \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$