Sampling and Reconstruction

Reading

- Required:
  - Watt, Section 14.1
- Recommended:
  - Ron Bracewell, The Fourier Transform and Its Applications, McGraw-Hill.
  - Don P. Mitchell and Arun N. Netravali, “Reconstruction Filters in Computer Computer Graphics ,” Computer Graphics, (Proceedings of SIGGRAPH 88). *22 (4)*, pp. 221-228, 1988.

What is an image?

- We can think of an **image** as a function, f, from R^2 to R:
  - $f(x, y)$ gives the intensity of a channel at position (x, y)
  - Realistically, we expect the image only to be defined over a rectangle, with a finite range:

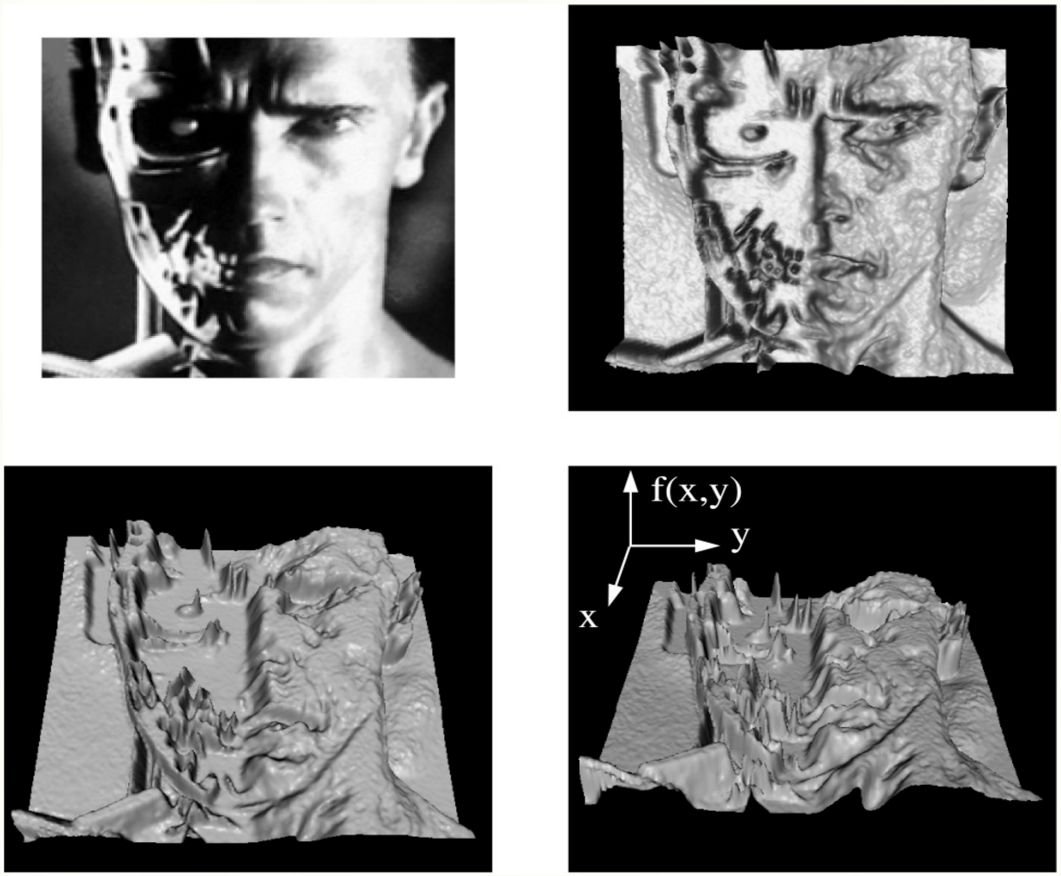
$$f: [a,b] \times [c,d] \rightarrow [0,1]$$

- A color image is just three functions pasted together. We can write this as a "vector-valued" function:

$$f(x,y) = \begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix}$$

- We'll focus in grayscale (scalar-valued) images for now.

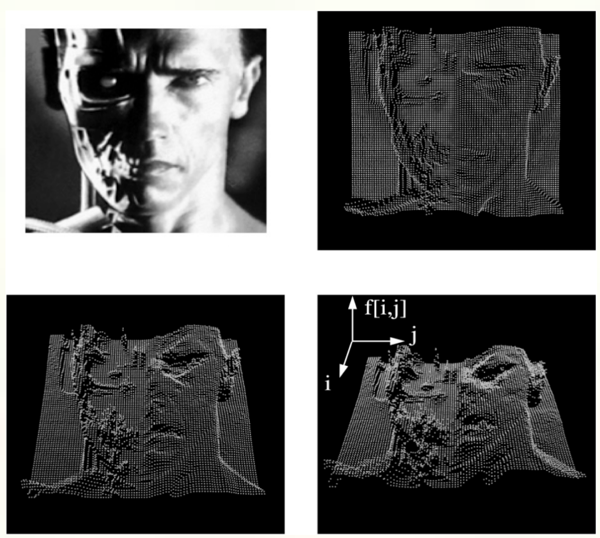
Images as functions



Digital images

- In computer graphics, we usually create or operate on **digital (discrete)** images:
  - **Sample** the space on a regular grid
  - **Quantize** each sample (round to nearest integer)
- If our samples are Δ apart, we can write this as:

$$f[i\ ,j] = \text{Quantize}\{\ f(i\ \Delta, j\ \Delta)\ \}$$



Motivation: filtering and resizing

- What if we now want to:
  - smooth an image?
  - sharpen an image?
  - enlarge an image?
  - shrink an image?
- Before we try these operations, it's helpful to think about images in a more mathematical way…

Fourier transforms

- We can represent a function as a linear combination (weighted sum) of sines and cosines.
- We can think of a function in two complementary ways:
  - **Spatially** in the **spatial domain**
  - **Spectrally** in the **frequency domain**
- The **Fourier transform** and its inverse convert between these two domains:

Spatial domain

$$\longrightarrow \quad F(s) = \int_{-\infty}^{\infty} f(x) e^{-i 2\pi sx} dx \quad \longrightarrow$$

$$\longleftarrow \quad f(x) = \int_{-\infty}^{\infty} F(s) e^{i 2\pi sx} ds \quad \longleftarrow$$

Frequency domain

Fourier transforms (cont'd)

Spatial domain → $F(s)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi sx}dx$ → Frequency domain

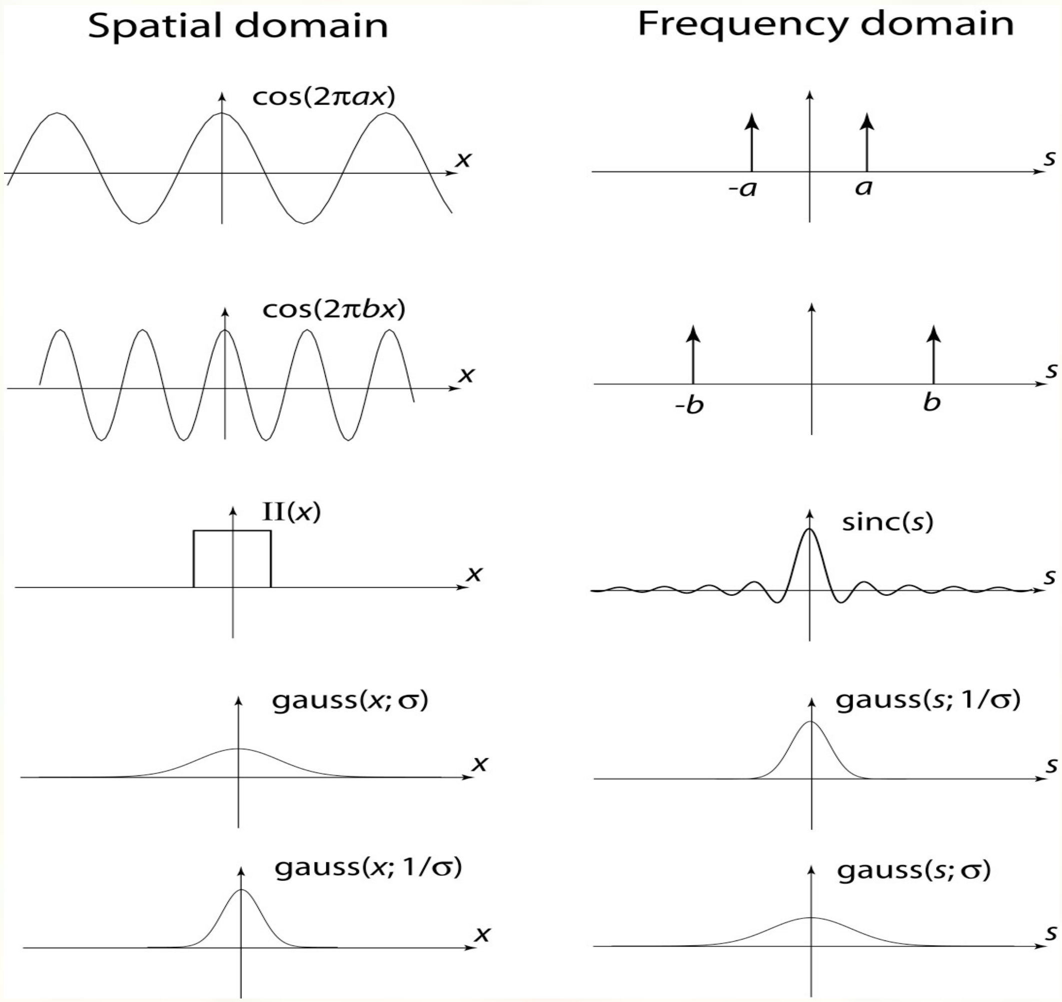
Spatial domain ← $f(x)=\int_{-\infty}^{\infty} F(s)e^{i2\pi sx}ds$ ← Frequency domain

- Where do the sines and cosines come in?
- $f(x)$ is usually a real signal, but $F(s)$ is generally complex:

$$F(s)=\mathrm{Re}(s)-i\,\mathrm{Im}(s)=|F(s)|e^{-i2\pi s}$$

- If $f(x)$ is symmetric, i.e., $f(x) = f(-x)$), then $F(s) = \mathrm{Re}(s)$. Why?

1D Fourier examples



2D Fourier transform

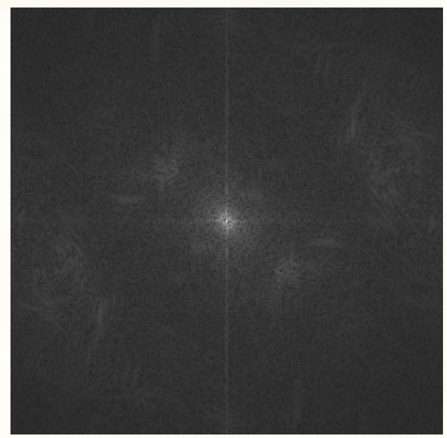
Spatial domain → $F(s_x, s_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-i2\pi s_x x} e^{-is\pi s_y y} \, dx dy$ → Frequency domain

Spatial domain ← $f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s_x, s_y) e^{i2\pi s_x x} e^{i2\pi s_y y} \, ds_x ds_y$ ← Frequency domain

Spatial domain

$f(x,y)$

Frequency domain

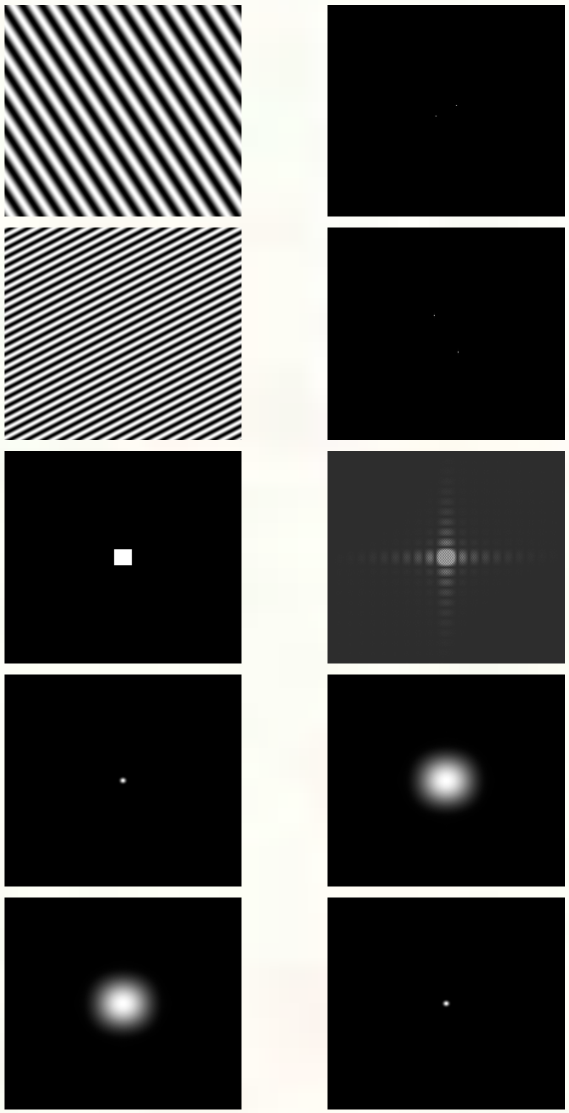


$\left| F(s_x, s_y) \right|$

2D Fourier examples

Spatial domain

$f(x,y)$



Frequency domain

$|F(s_x,s_y)|$

Convolution

- One of the most common methods for filtering a function is called **convolution**.
- In 1D, convolution is defined as:

$$
\begin{aligned}
g(x) &= f(x) * h(x) \\
&= \int_{-\infty}^{\infty} f(x')h(x - x')dx' \\
&= \int_{-\infty}^{\infty} f(x')\widehat{h}(x' - x)dx'
\end{aligned}
$$

where $\widehat{h}(x) = h(-x)$

Convolution properties

- Convolution exhibits a number of basic, but important properties.
- Commutativity: $a(x) * b(x) = b(x) * a(x)$
- Associativity: $[a(x) * b(x)] * c(x) = a(x) * [b(x) * c(x)]$
- Linearity: $a(x) * [k \cdot b(x)] = k \cdot [a(x) * b(x)]$

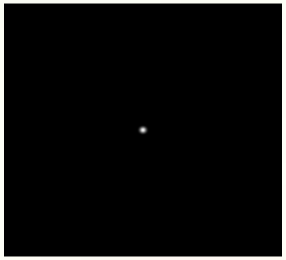
$$a(x) * (b(x) + c(x)) = a(x) * b(x) + a(x) * c(x)$$

Convolution in 2D

- In two dimensions, convolution becomes:

$$g(x,y) = f(x,y) * h(x,y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x')(y-y')dx'dy'$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\hat{h}(x'-x)(y'-y)dx'dy'$$

where $\hat{h}(x,y) = h(-x,-y)$

 *  = 

$f(x,y)$ $h(x,y)$ $g(x,y)$

Convolution theorems

- **Convolution theorem**: *Convolution* in the *spatial* domain is equivalent to *multiplication* in the *frequency* domain.

$$f * h \Leftrightarrow F \cdot H$$

- **Symmetric theorem**: *Convolution* in the *frequency* domain is equivalent to *multiplication* in the *spatial* domain.

$$f \cdot h \Leftrightarrow F * H$$

Convolution theorems

Theorem

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g)$$

$$\mathcal{F}^{-1}(F * G) = \mathcal{F}^{-1}(F)\mathcal{F}^{-1}(G)$$

$$\mathcal{F}^{-1}(FG) = \mathcal{F}^{-1}(F) * \mathcal{F}^{-1}(G)$$

Proof (1)

$$\mathcal{F}(f * g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t')g(t-t')dt'e^{-i\omega t}dt$$

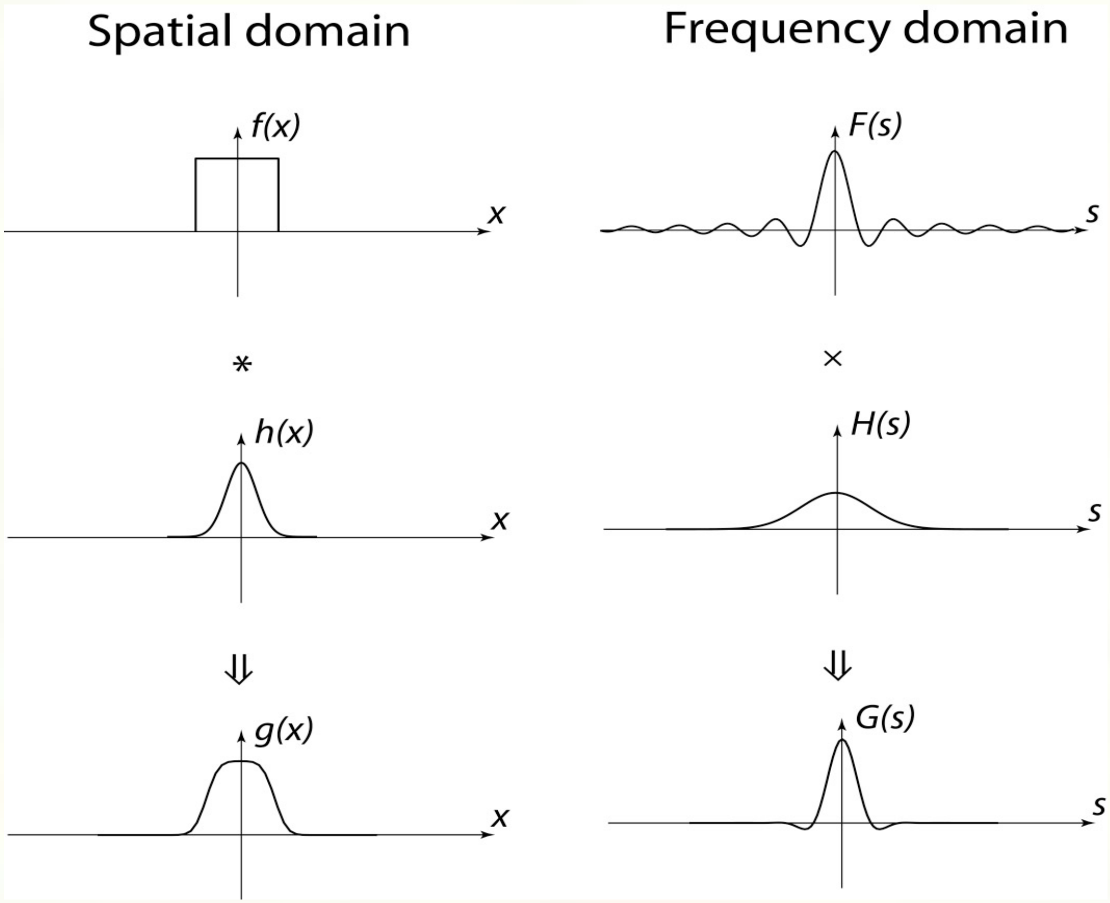
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t')g(t-t')e^{-i\omega t'}e^{-i\omega(t-t')}dtdt'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t')g(t'')e^{-i\omega t'}e^{-i\omega t''}dt''dt'$$

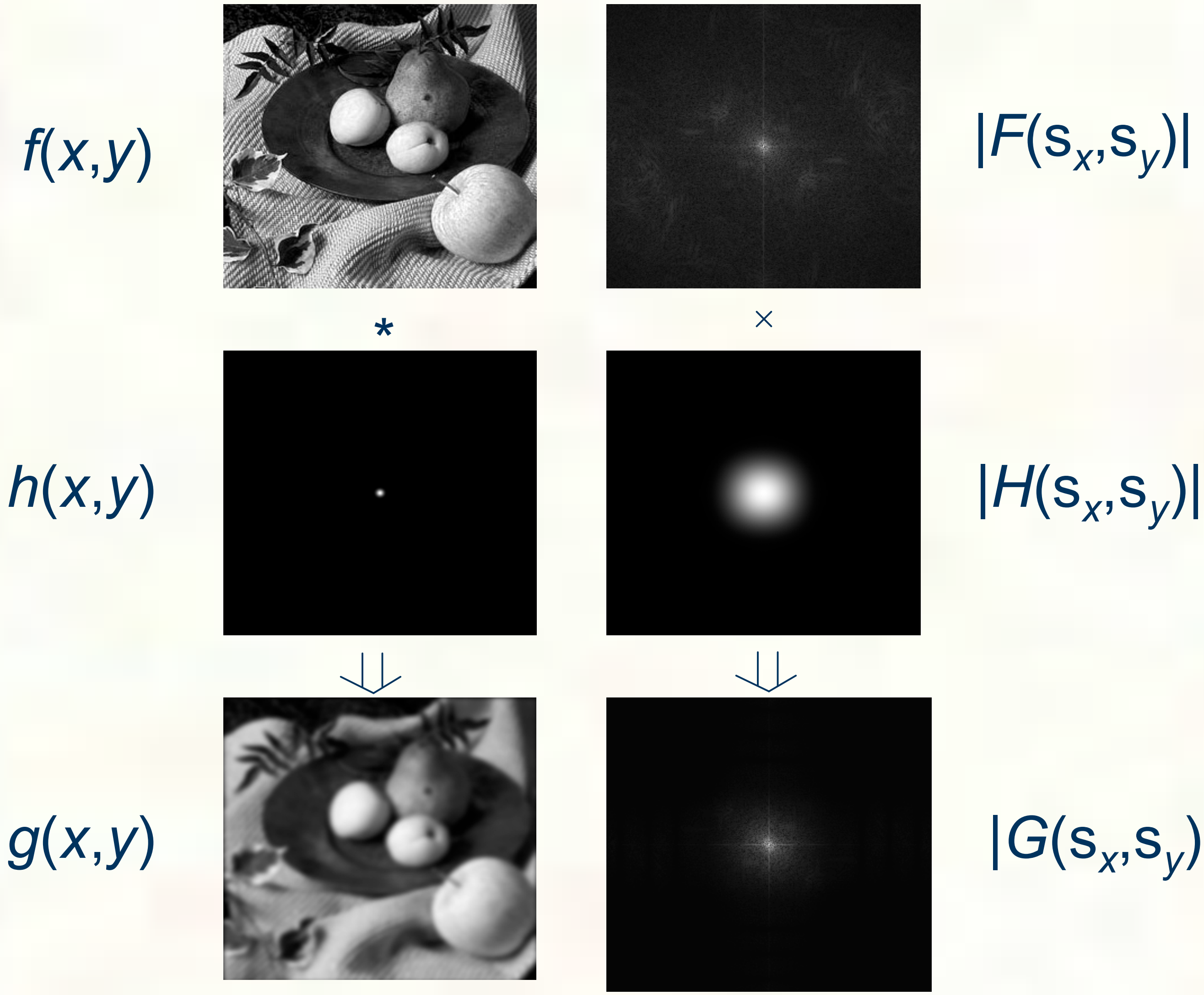
$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\int_{-\infty}^{\infty} g(t'')e^{-i\omega t''}dt''$$

$$= \mathcal{F}(f)\mathcal{F}(g)$$

1D convolution theorem example

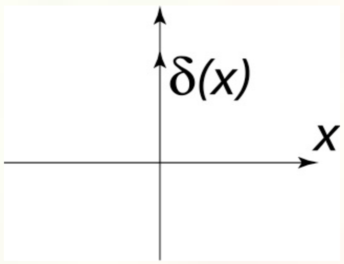


2D convolution theorem example



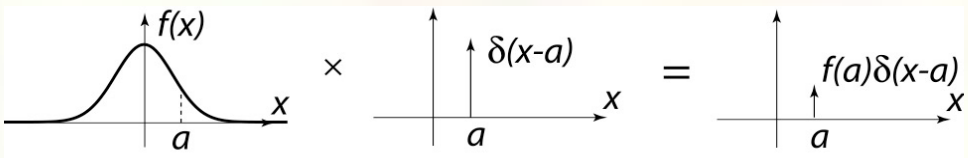
The delta function

- The **Dirac delta function**, $\delta(x)$, is a handy tool for sampling theory.
- It has zero width, infinite height, and unit area.
- It is usually drawn as:

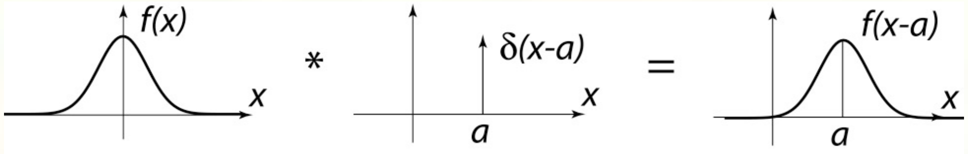


Sifting and shifting

- For sampling, the delta function has two important properties.
- **Sifting**: $f(x)\delta(x-a) = f(a)\delta(x-a)$



- **Shifting**: $f(x) * \delta(x-a) = f(x-a)$

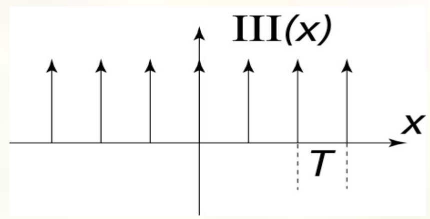


The shah/comb function

- A string of delta functions is the key to sampling. The resulting function is called the **shah** or **comb** function:

$$\mathrm{III}(x) = \sum_{n=0}^{\infty} \delta(x - nT)$$

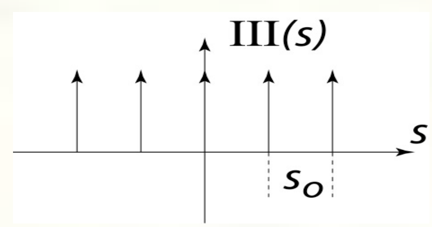
- which looks like:



- Amazingly, the Fourier transform of the shah function takes the same form:

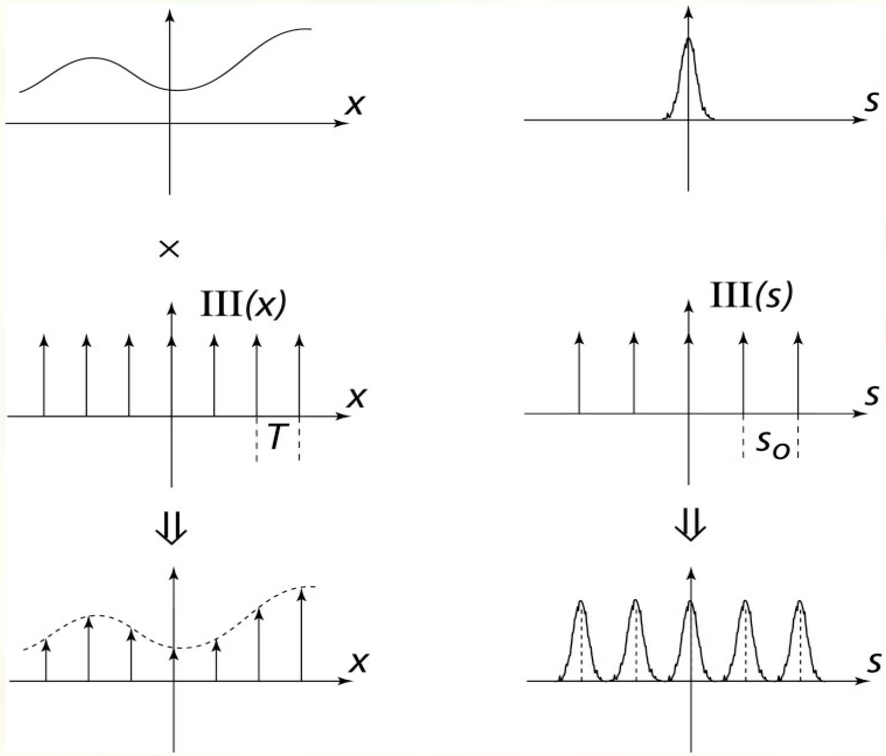
$$\mathrm{III}(s) = \sum_{n=0}^{\infty} \delta(s - ns_0)$$

where $s_o = 1/T$.



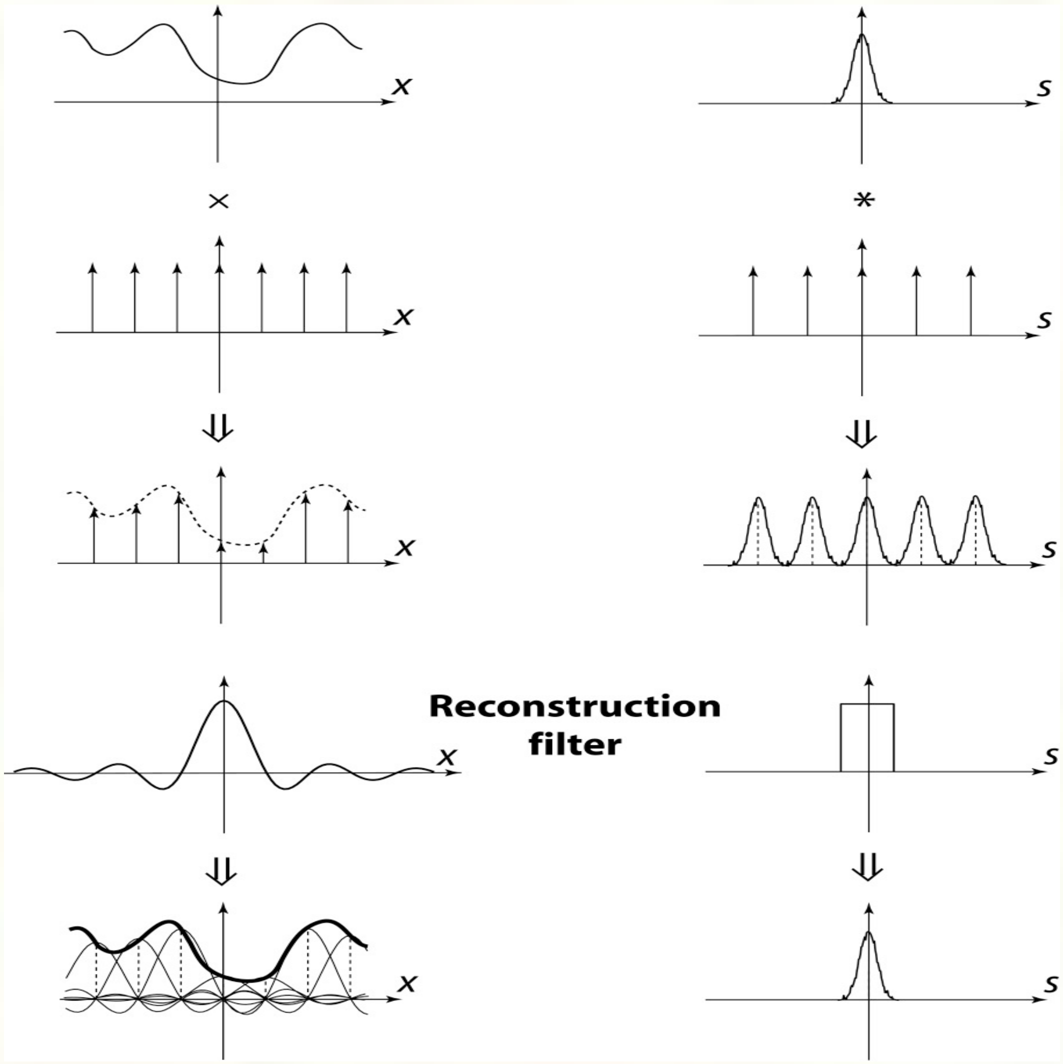
Sampling

- *Now*, we can talk about sampling.

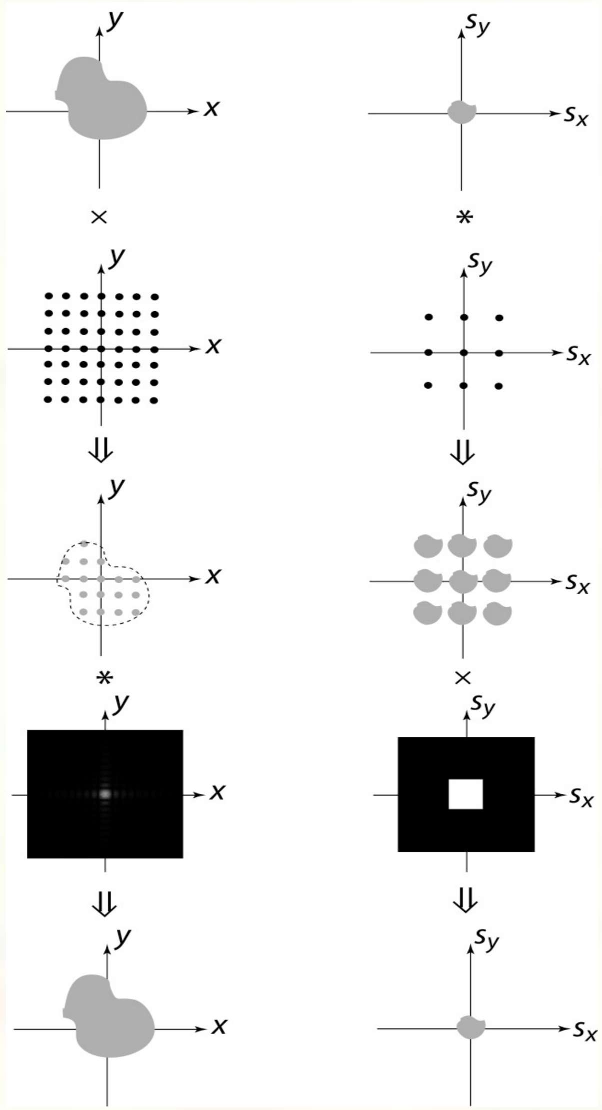


- The Fourier spectrum gets *replicated* by spatial sampling!
- How do we recover the signal?

Sampling and reconstruction



Sampling and reconstruction in 2D



Sampling theorem

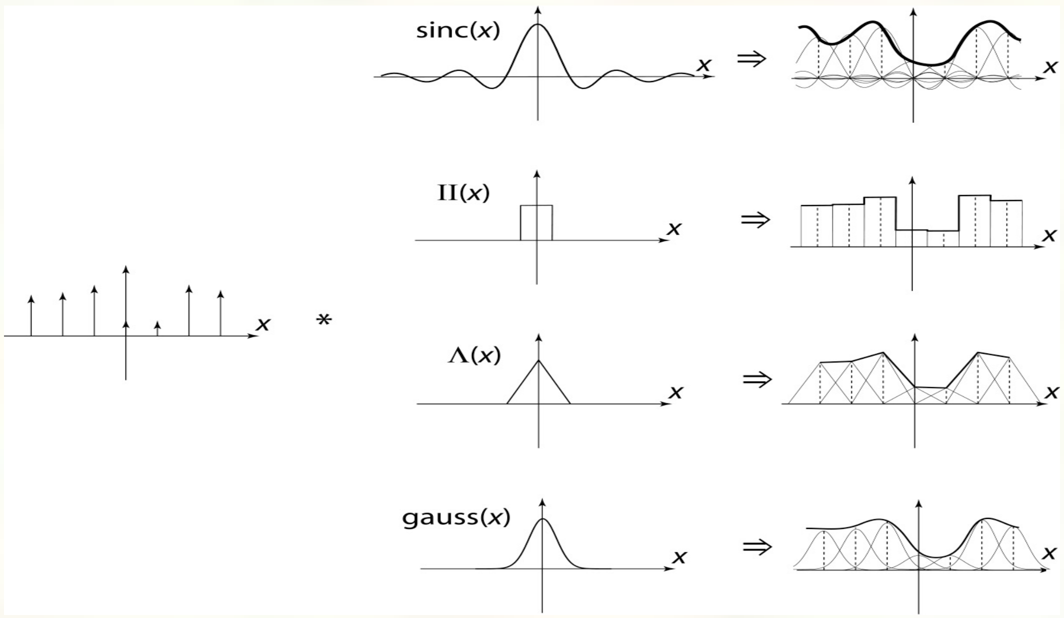
- This result is known as the **Sampling Theorem** and is generally attributed to Claude Shannon (who discovered it in 1949) but was discovered earlier, independently by at least 4 others:

  A signal can be reconstructed from its samples without loss of information, if the original signal has no energy in frequencies at or above ½ the sampling frequency.

- For a given **bandlimited** function, the minimum rate at which it must be sampled is the **Nyquist frequency**.

Reconstruction filters

- The sinc filter, while "ideal", has two drawbacks:
  - It has large support (slow to compute)
  - It introduces ringing in practice
- We can choose from many other filters…

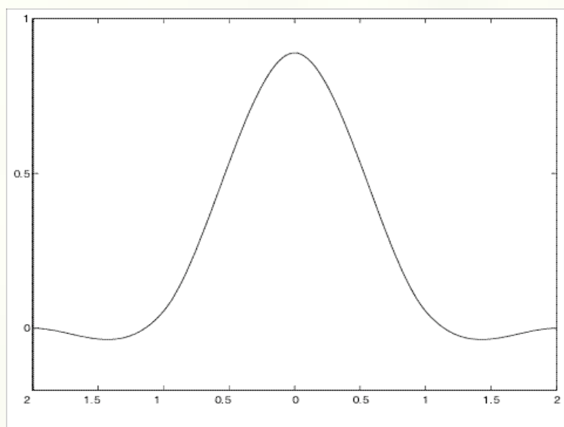


Cubic filters

- Mitchell and Netravali (1988) experimented with cubic filters, reducing them all to the following form:

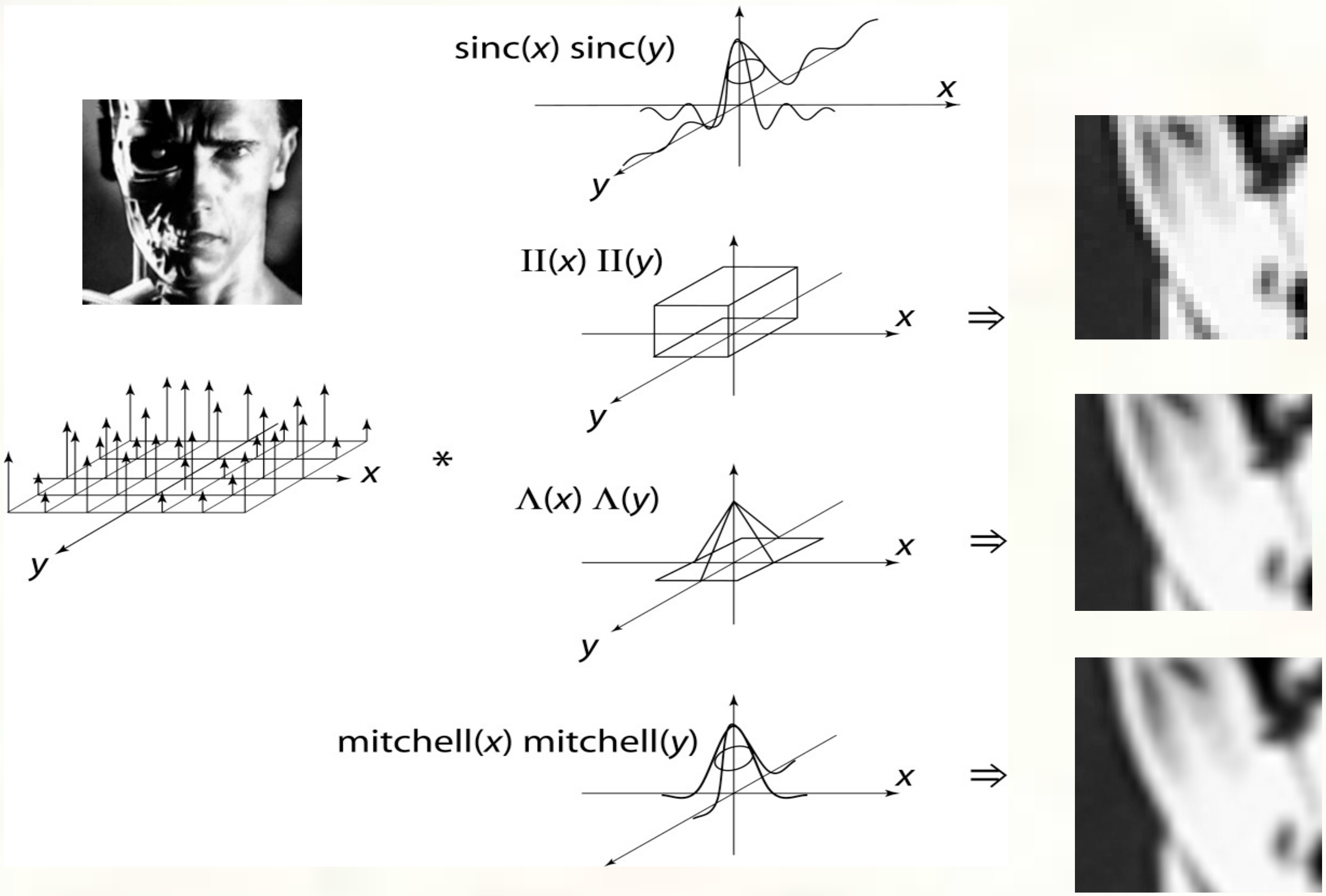
$$r(x)=\frac{1}{6}\begin{cases}(12-9B-6C)|x|^3+(-18+12B+6C)|x|^2+(6-2B) & |x|<1\\ ((-B-6C)|x|^3+(6B+30C)|x|^2+(-12B-48C)|x|+(8B+24C) & 1\le|x|<2\\ 0 & \mathit{otherwise}\end{cases}$$

- The choice of B or C trades off between being too blurry or having too much ringing. B=C=1/3 was their "visually best" choice.
- The resulting reconstruction filter is often called the "Mitchell filter."

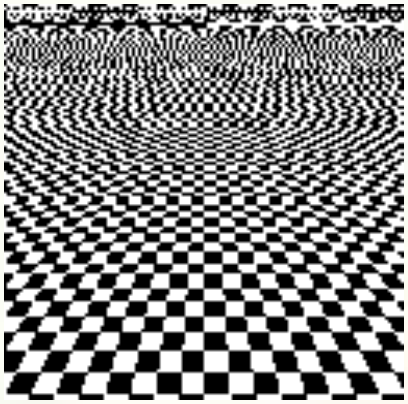


Reconstruction filters in 2D

- We can also perform reconstruction in 2D…



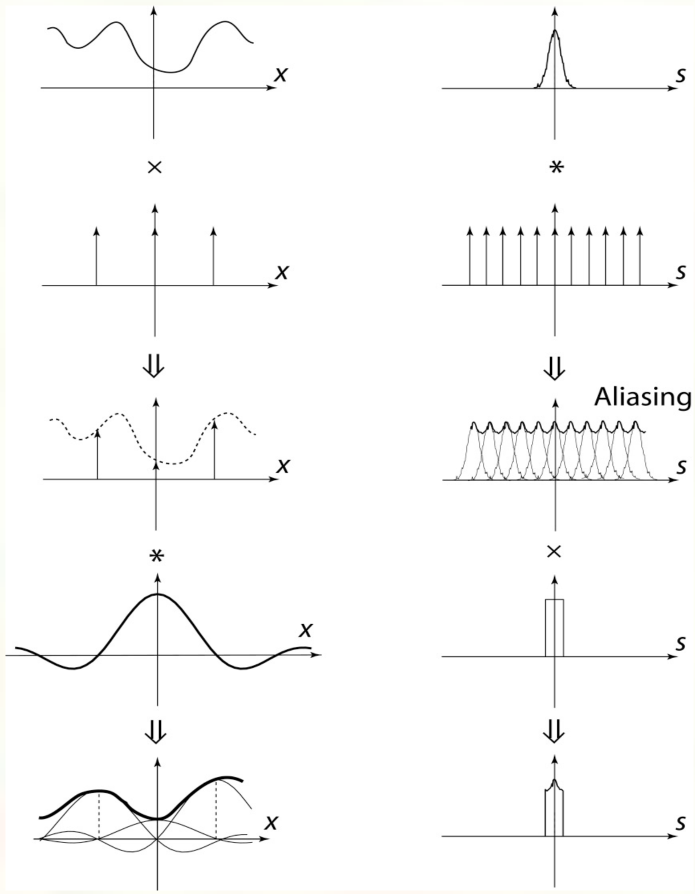
Aliasing



Sampling rate is too low

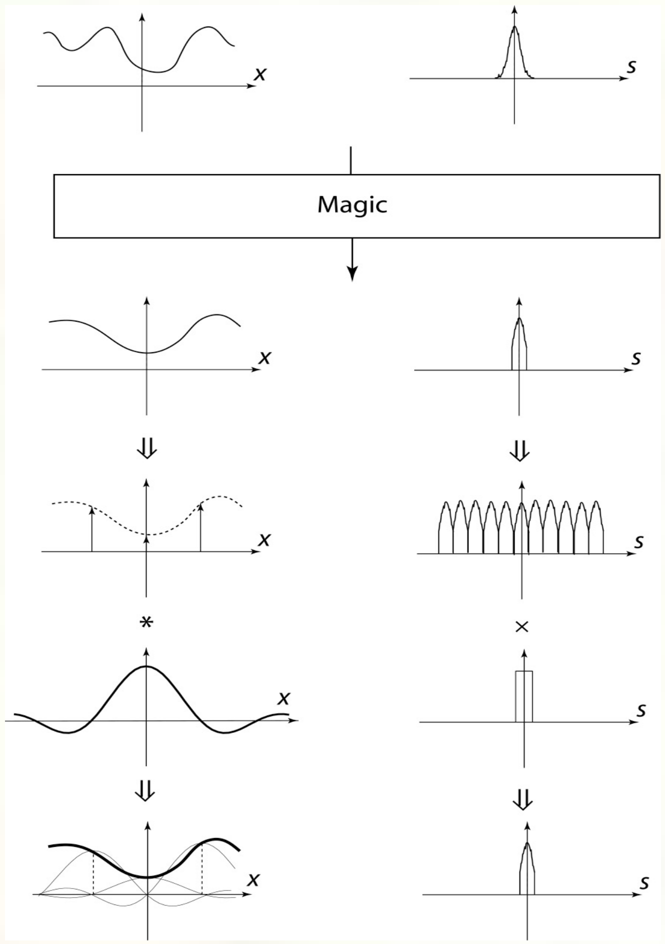
Aliasing

- What if we go below the Nyquist frequency?



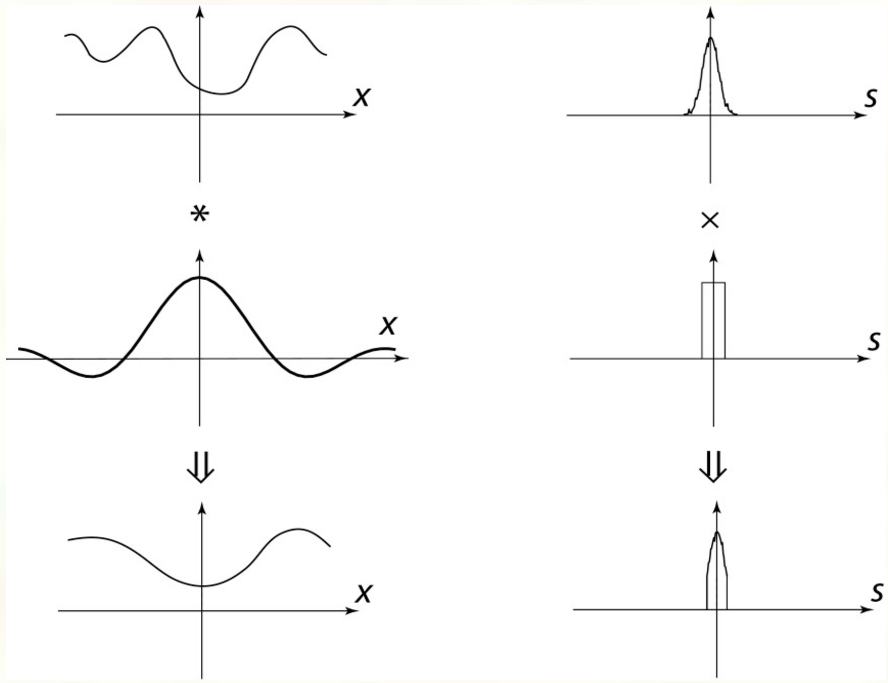
Anti-aliasing

- **Anti-aliasing** is the process of *removing* the frequencies before they alias.



Anti-aliasing by analytic prefiltering

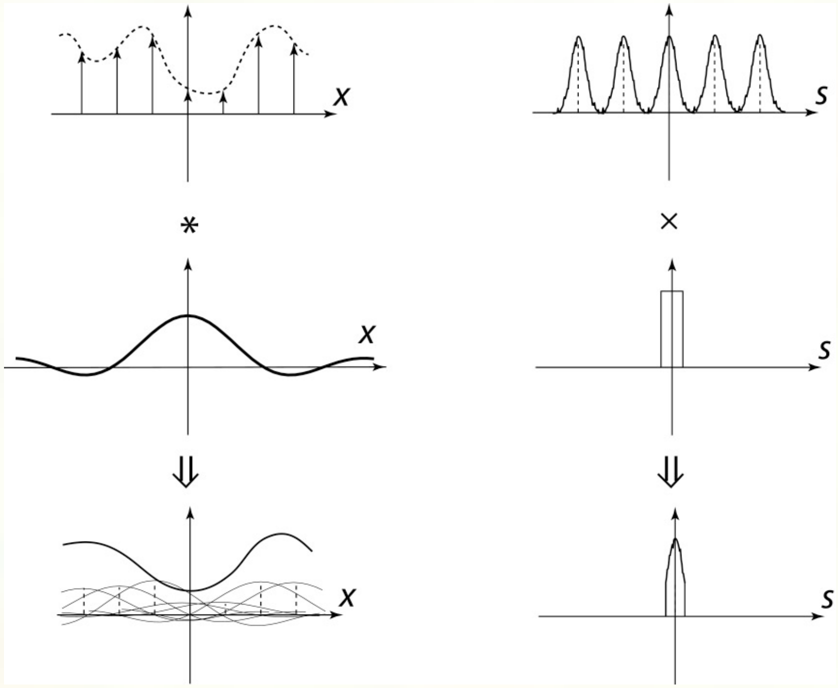
- We can fill the "magic" box with analytic pre-filtering of the signal:



- Why may this not generally be possible?

Filtered downsampling

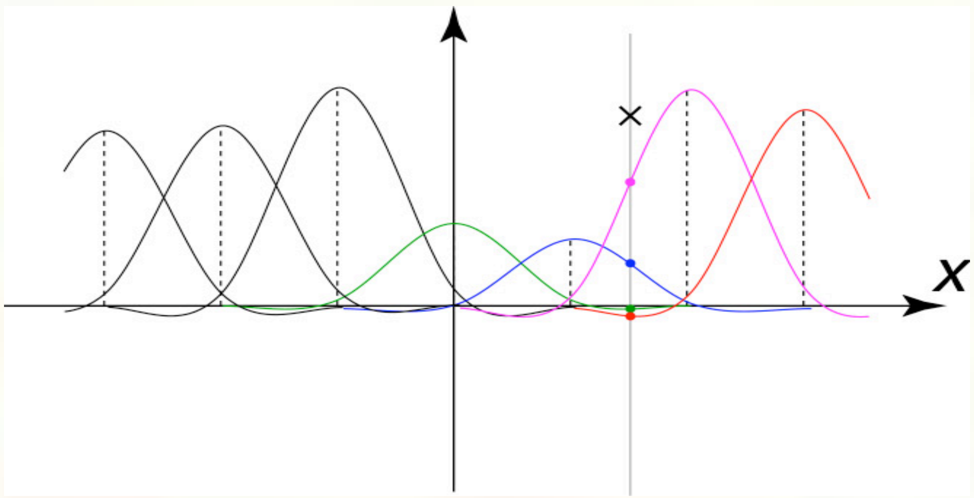
- Alternatively, we can sample the image at a higher rate, and then filter that signal:



- We can now sample the signal at a lower rate. The whole process is called filtered **downsampling** or **supersampling and averaging down**.

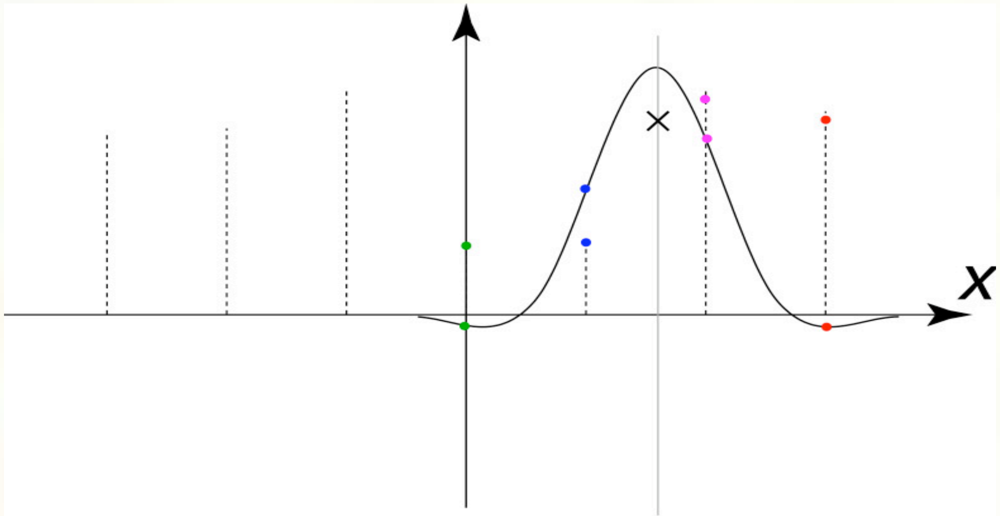
Practical upsampling

- When resampling a function (e.g., when resizing an image), you do not need to reconstruct the complete continuous function.
- For zooming in on a function, you need only use a reconstruction filter and evaluate as needed for each new sample.
- Here's an example using a cubic filter:



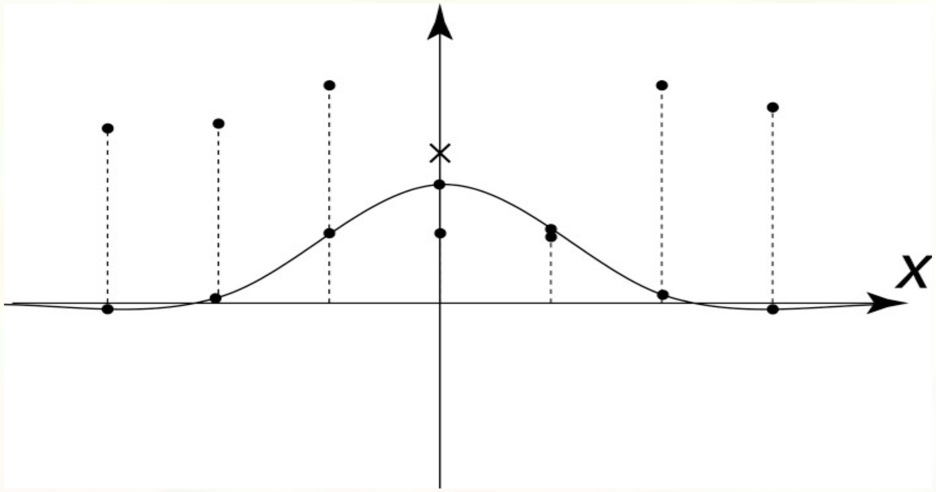
Practical upsampling

- This can also be viewed as:
  1. putting the reconstruction filter at the desired location
  2. evaluating at the original sample positions
  3. taking products with the sample values themselves
  4. summing it up



Practical downsampling

- Downsampling is similar, but filter has larger support and smaller amplitude.
- Operationally:
  1. Choose filter in downsampled space.
  2. Compute the downsampling rate, *d*, ratio of new sampling rate to old sampling rate
  3. Stretch the filter by 1/*d* and scale it down by *d*
  4. Follow upsampling procedure (previous slides) to compute new values



2D resampling

We've been looking at **separable** filters:

$$r_{2D}(x,y) = r_{1D}(x)r_{1D}(y)$$

How might you use this fact for efficient resampling in 2D?

Next class: Image Processing

- Reading:
- Jain, Kasturi, Schunck, Machine Vision. McGraw-Hill, 1995.
  Sections 4.2-4.4, 4.5(intro), 4.5.5, 4.5.6, 5.1-5.4.
  (from course reader)

- Topics:
  - Implementing discrete convolution
  - Blurring and noise reduction
  - Sharpening
  - Edge detection