Subdivision curves

Reading

- Recommended:
  - Stollnitz, DeRose, and Salesin. *Wavelets for Computer Graphics: Theory and Applications*, 1996, section 6.1-6.3, A.5.

- Note: there is an error in Stollnitz, et al., section A.5. Equation A.3 should read:
  - $\mathbf{MV} = \mathbf{V\Lambda}$

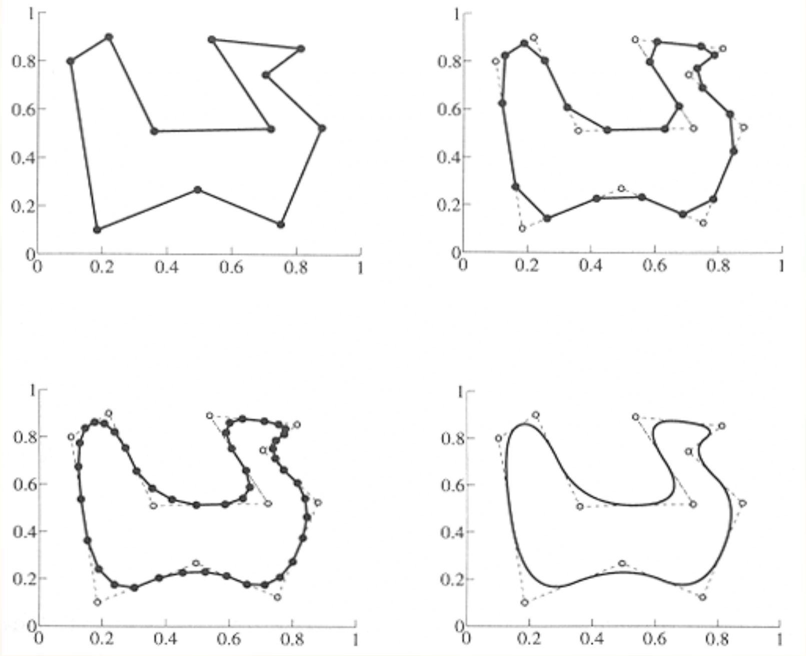
Subdivision curves

- Idea:
  - repeatedly refine the control polygon

$$P^1 \rightarrow P^2 \rightarrow P^2 \rightarrow \cdots$$

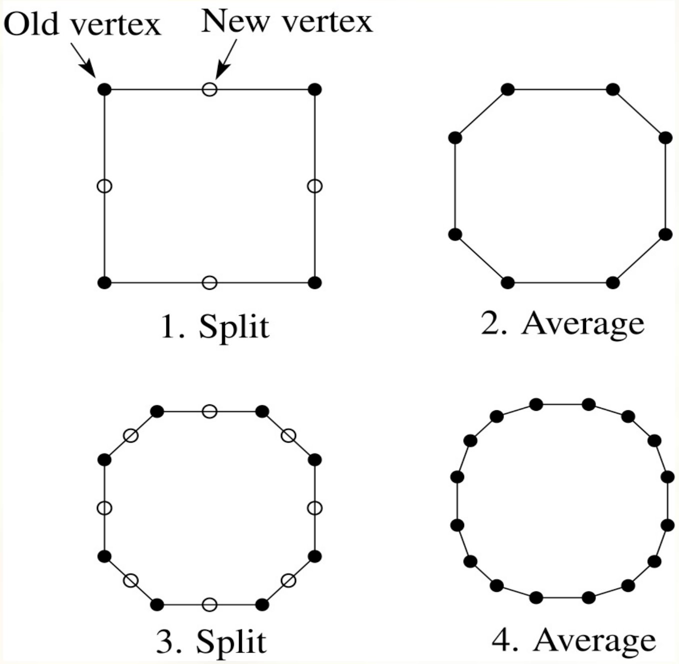
  - curve is the limit of an infinite process

$$Q = \lim_{j \rightarrow \infty} P^j$$



Chaikin’s algorithm

- Chaikin introduced the following “corner-cutting” scheme in 1974:
  - Start with a piecewise linear curve
  - Insert new vertices at the midpoints (the **splitting step**)
  - Average each vertex with the “next” (clockwise) neighbor (the **averaging step**)
  - Go to the splitting step



Averaging masks

- The limit curve is a quadratic B-spline!
- Instead of averaging with the nearest neighbor, we can generalize by applying an **averaging mask** during the averaging step:

$$r = (\ldots, r_{-1}, r_0, r_1, \ldots)$$

- In the case of Chaikin's algorithm:

$$r = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Can we generate other B-splines?

- Answer: Yes   Lane-Riesenfeld algorithm (1980)
- Use averaging masks from Pascal's triangle:

$$r = \frac{1}{2^n}\left(\binom{n}{0},\binom{n}{1},\cdots,\binom{n}{n}\right)$$

- Gives B-splines of degree *n*+1.
- n=0: 1

- n=1: 1

1 1

- n=2: 1

1 1

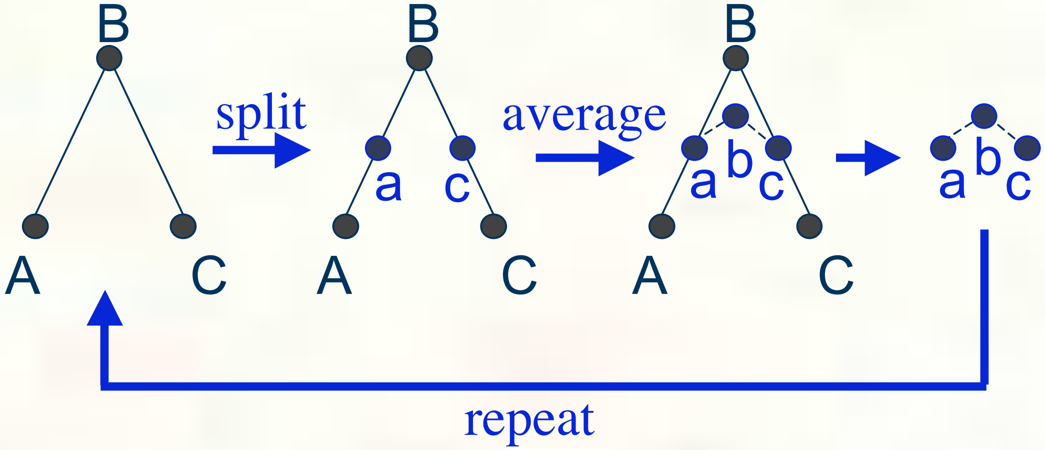
1 2 1

Subdivide ad nauseum?

- After each split-average step, we are closer to the **limit curve**.
- How many steps until we reach the final (limit) position?
- Can we push a vertex to its limit position without infinite subdivision? Yes!

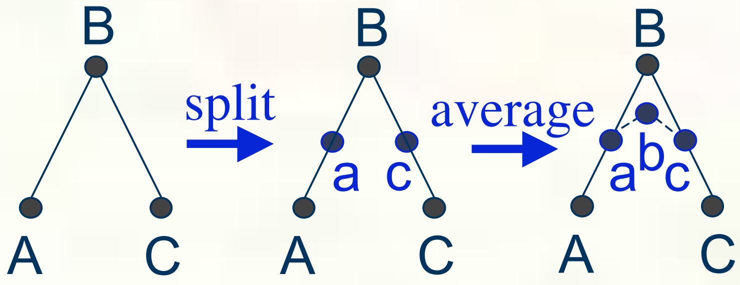
One subdivision step

- Consider the cubic B-spline subdivision mask: $\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$
- Now consider what happens during splitting and averaging:



Math for one subdivision step

- Subdivision mask: $\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$
- One subdivision step:



Split: $\mathbf{a} = \frac{1}{2}(\mathbf{A}+\mathbf{B})$

$\mathbf{c} = \frac{1}{2}(\mathbf{B}+\mathbf{C})$

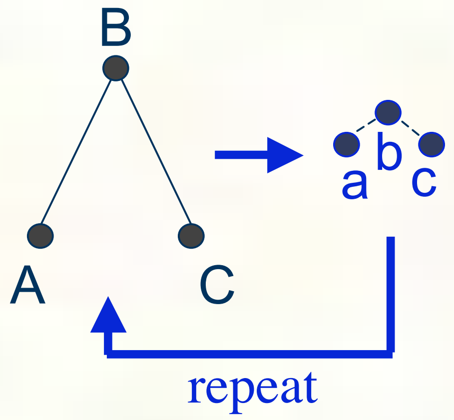
Average:

a and **c** do not change

$$\mathbf{b} = \frac{1}{4}(\mathbf{a}+2B+\mathbf{c}) = \frac{1}{8}(\mathbf{A}+6\mathbf{B}+\mathbf{C})$$

Consolidated math for one step

- Subdivision mask: $\frac{1}{4}\begin{pmatrix}1 & 2 & 1\end{pmatrix}$
- One subdivision step:



- Consolidated math for one subdivision step:

$$\underbrace{\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{bmatrix}}_{P_{j+1}} = \frac{1}{8}\underbrace{\begin{bmatrix}4 & 4 & 0\\ 1 & 6 & 1\\ 0 & 4 & 4\end{bmatrix}}_{\text{Local subdivision matrix 'S'}}\underbrace{\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{bmatrix}}_{P_j}$$

Local subdivision matrix, cont’d

- Tracking the point’s value through subdivision:

$$P_j = SP_{j-1} = S \cdot SP_{j-2} = S \cdot S \cdot SP_{j-3} = \cdots = S^j P_0$$

- The limit position of the point is then:

$$P_\infty = S^\infty P_0$$

- or as we’d say in calculus…

$$P_\infty = \lim_{j \to \infty} S^j P_0$$

- OK, so how do we apply a matrix an infinite number of times??

Eigenvectors and eigenvalues

- To solve this problem, we need to look at the eigenvectors and eigenvalues of **S**. First, a review…
  - Let v be a vector such that $\mathbf{S}\mathbf{v} = \lambda\mathbf{v}$
  - We say that **v** is an eigenvector with eigenvalue λ.
  - An n x n matrix can have n eigenvalues and eigenvectors:

$$\mathbf{S}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$$
$$\vdots$$
$$\mathbf{S}\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

  - If the eigenvectors are linearly independent (which means that **S** is *non-defective*), then they form a basis, and we can re-write P in terms of the eigenvectors:

$$P = \sum_{i=1}^{n} a_i\mathbf{v}_i$$

To infinity, but not beyond…

- So, applying S to P:

$$P_1 = SP_0 = S\sum_i^n a_i v_i = \sum_i^n a_i S v_i = \sum_i^n a_i \lambda_i v_i$$

- Applying it *j* times:

$$P_j = S^j P_0 = S^j \sum_i^n a_i v_i = \sum_i^n a_i S^j v_i = \sum_i^n a_i \lambda_i^j v_i$$

- Let's assume the eigenvalues are non-negative and sorted so that:

$$\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

- Now let *j* go to infinity: $P_\infty = \lim_{j\to\infty} S^j P_0 = \lim_{j\to\infty} \sum_i^n a_i \lambda_i^j v_i$

- If $\lambda_1 > 1$, then:
- If $\lambda_1 < 1$, then:
- If $\lambda_1 = 1$, then:

Evaluation masks

- What are the eigenvalues and eigenvectors of our cubic B-spline subdivision matrix?

$$\lambda_1 = 1 \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_3 = \frac{1}{4} \quad V_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

- We're OK!
- But what is the final position?

$$P_\infty = \lim_{j \to \infty} \left(a_1 \lambda_1^j v_1 + a_2 \lambda_2^j v_2 + a_3 \lambda_3^j v_3 \right)$$

$$P_\infty =$$

- Almost done… from earlier we know that we can find 'a_i', we but didn't give specifics.

Evaluation masks, cont’d

- To finish up, we need to compute a_1.
- Remember: $$P_0 = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$
- Rewrite as: $$P_0 = \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{V}A$$
- We need to solve for the vector ‘A’.

  (This is really just a change of basis for representing the vector P).

  $$A = \mathbf{V}^{-1} P_0$$

  The solution is:

  $$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \cdots & u_1^T & \cdots \\ \cdots & u_2^T & \cdots \\ & \vdots & \\ \cdots & u_n^T & \cdots \end{bmatrix} P_0$$

- Now we can compute the limit position: $$P_\infty = a_1 = u_1^T P_0$$
- We call u_1 the **evaluation mask**.

Evaluation masks, cont'd

- Note that we need not start with the 0th level control points and push them to the limit.
- If we subdivide and average the control polygon *j* times, we can push the vertices of the refined polygon to the limit as well:

$$P_\infty = S^\infty P_j = u_1^T P_j$$

- So far we've been looking at math for a subdivision function f(x).
- For a 2D parametric subdivision curve, (x(u), y(u)), just apply these formulas separately for the x(u) and y(u) functions.

Recipe for subdivision curves

- The evaluation mask for the cubic B-spline is:

$$\frac{1}{6}\begin{pmatrix}1 & 4 & 1\end{pmatrix}$$

- Now we can cook up a simple procedure for creating subdivision curves:
  - Subdivide (split+average) the control polygon a few times. Use the averaging mask.
  - Push the resulting points to the limit positions. Use the evaluation mask.

Derivative of subdiv. function

- What is the tangent to the cubic B-spline function?
- Consider the formula for P again:

$$P_j = a_1\lambda_1^j v_1 + a_2\lambda_2^j v_2 + a_3\lambda_3^j v_3$$

$$P_j = a_1(1)^j \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a_2(\frac{1}{2})^j \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + a_3(\frac{1}{4})^j \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Where:

$$P_j = \begin{bmatrix} left \\ center \\ right \end{bmatrix} \qquad P' = \lim_{\Delta x \to 0} \frac{center - left}{\Delta x} = \lim_{j \to \infty} \frac{center - left}{\frac{1}{2^j}}$$

Derivative is just: $P' = \lim_{j \to \infty} \left(a_2 \left(\frac{1}{2}\right)^j \frac{0+1}{\frac{1}{2^j}} \right) = a_2 = u_2^T P_0$

Tangent analysis for 2D curve

- What is the tangent to a parametric cubic B-spline **2D curve**?
- Using a similar derivation to what we just did for a 1D function (but omitting details):

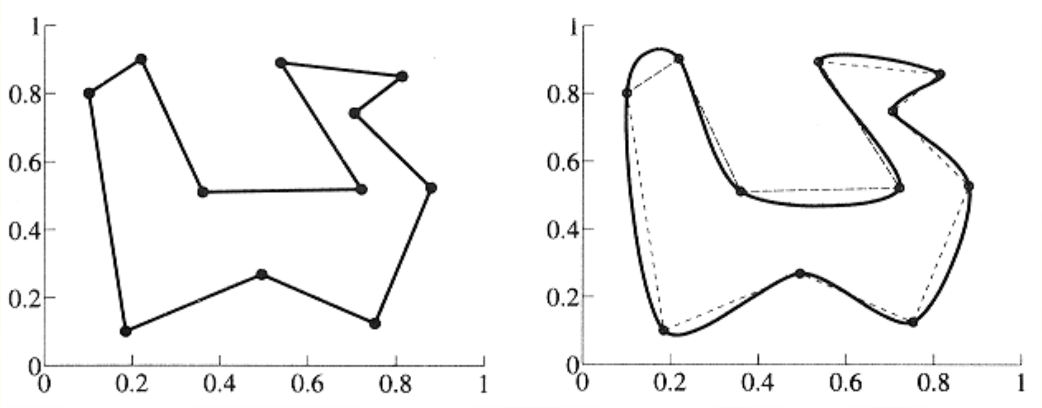
$$\mathbf{t} = \lim_{j \to \infty} \frac{P_{Center,j} - P_{Left,j}}{\left\| P_{Center,j} - P_{Left,j} \right\|} = \frac{u_2^T P_0}{\left\| u_2^T P_0 \right\|}$$

- Thus, we can compute the tangent using the *second* left eigenvector! This analysis holds for general subdivision curves and gives us the **tangent mask**.

Approximation vs. Interpolation of Control Points

- Previous subdivision scheme *approximated* control points. Can we *interpolate* them?
  Yes: **DLG interpolating scheme (1987)**
- Slight modification to subdivision algorithm:
  - splitting step introduces midpoints
  - averaging step *only changes midpoints*
- For DLG (Dyn-Levin-Gregory), use:

$$r = \frac{1}{16}(-2, 5, 10, 5, -2)$$



- Since we are only changing the midpoints, the points after the averaging step do not move.