Interpolating curves

Reading

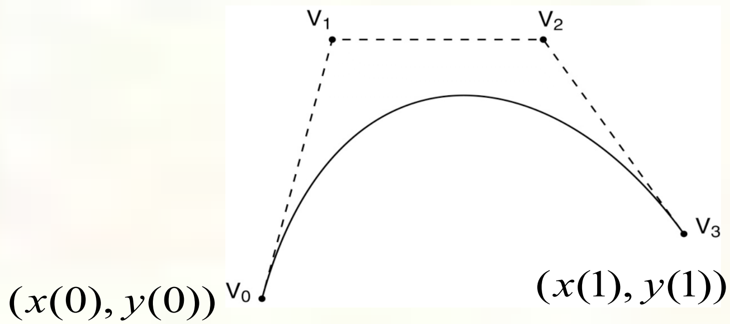
- Optional
  - Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987. (See course reader.)

Parametric curve review

Parametric curves

- We use parametric curves, *Q(u)=(x(u),y(u))*, where *x(u)* and *y(u)* are cubic polynomials:

$$x(u) = Au^3 + Bu^2 + Cu + D$$
$$y(u) = Eu^3 + Fu^2 + Gu + H$$



- Advantages:
  - easy (and efficient) to compute
  - "well behaved"
  - infinitely differentiable
- We also assume that *u* varies from 0 to 1

Various ways to set A,B,C,D

$$x(u) = Au^3 + Bu^2 + Cu + D$$

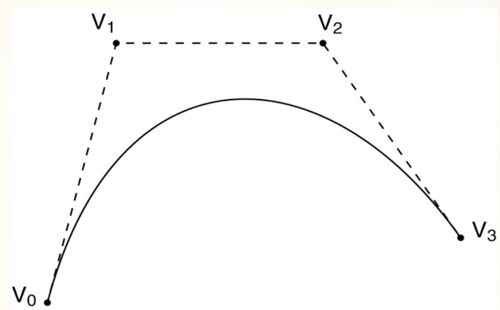
0) Directly – non-intuitive; not very useful.

1) Set positions and derivatives of endpoints:
"Hermite Curve"

2) Use "control points" that indirectly influence the curve:

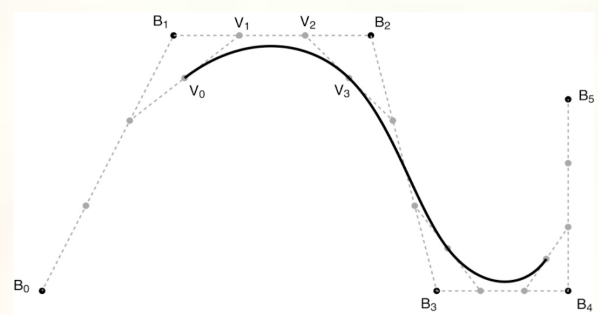
"Bezier curve":

- interpolates endpoints
- does not interpolate middle control points

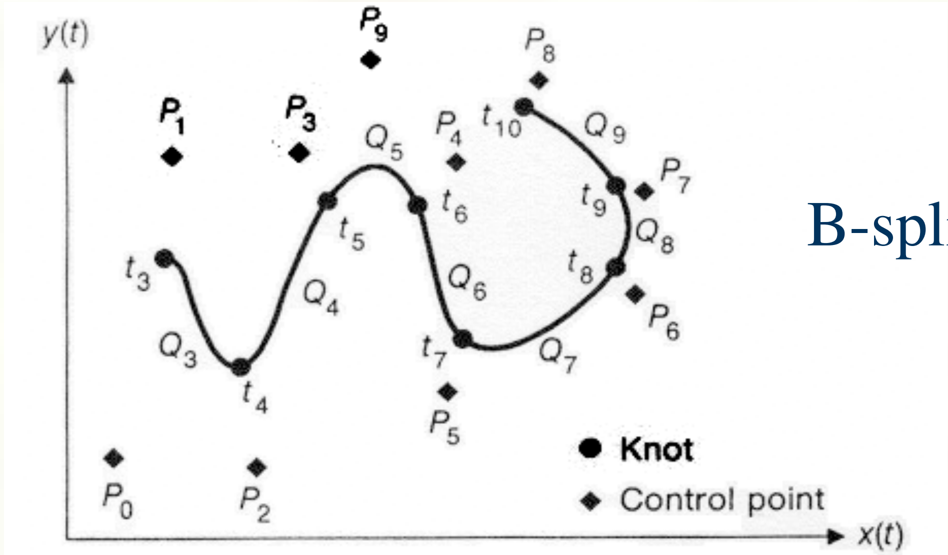


"B-spline"

- does not interpolate ANY control points



Splines = join cubic curves



B-spline

<u>Considerations</u>

What kind of continuity at join points ("knots")?

C0 = value
C1 = first derivative
C2 = second derivative

How do control points work?

Spline summary

- Joined Hermite curves:
  - C1 continuity
  - Interpolates control points
- B-splines:
  - C2 continuity
  - Does not interpolate control points

Can we get…

C2 continuity

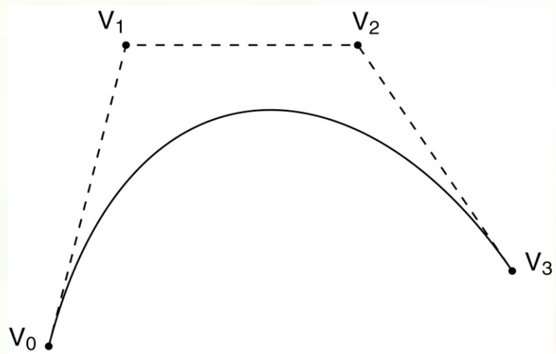
Interpolates control points

That's what we'll talk about towards the end of this lecture.
But first, some other useful tips.

Useful tips for Bézier curves

Displaying Bézier curves

- How could we draw one of these things?

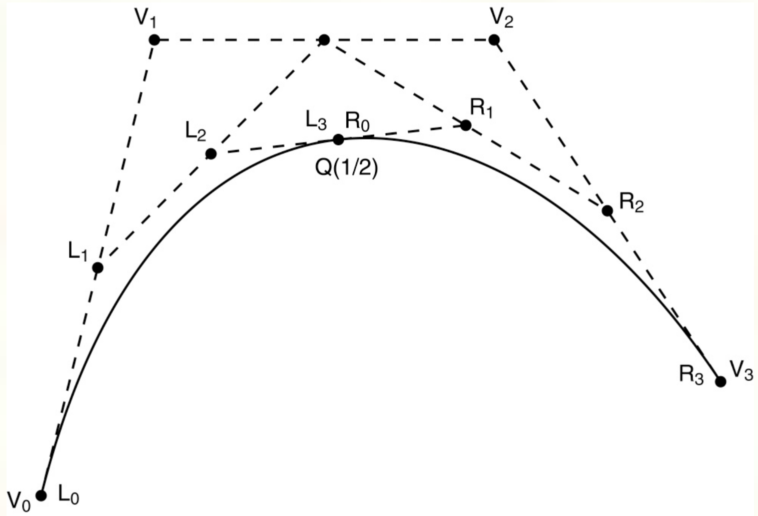


- It would be nice if we had an *adaptive* algorithm, that would take into account flatness.

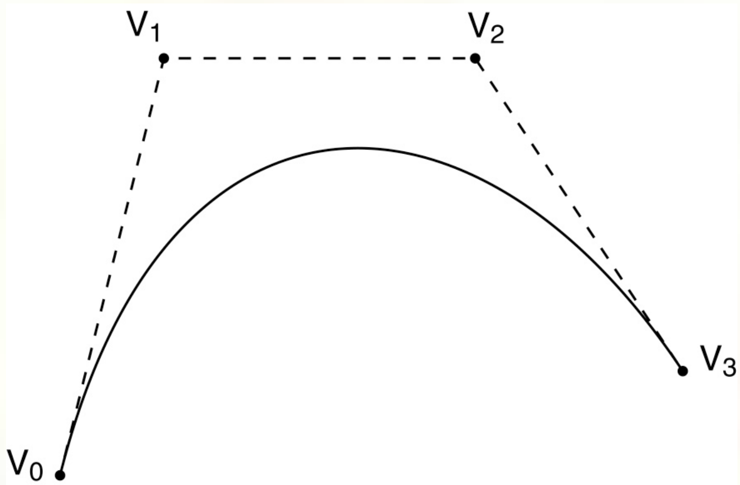
```
DisplayBezier( V0, V1, V2, V3 )
begin
    if ( FlatEnough( V0, V1, V2, V3 ) )
        Line( V0, V3 );
    else
        something;
end;
```

Subdivide and conquer

```
DisplayBezier( V0, V1, V2, V3 )
begin
    if ( FlatEnough( V0, V1, V2, V3 ) )
        Line( V0, V3 );
    else
        Subdivide(V[]) ⇒ L[], R[]
        DisplayBezier( L0, L1, L2, L3 );
        DisplayBezier( R0, R1, R2, R3 );
end;
```



Testing for flatness



Compare total length of control polygon to length of line connecting endpoints:

$$\frac{|V_0 - V_1| + |V_1 - V_2| + |V_2 - V_3|}{|V_0 - V_3|} < 1 + \varepsilon$$

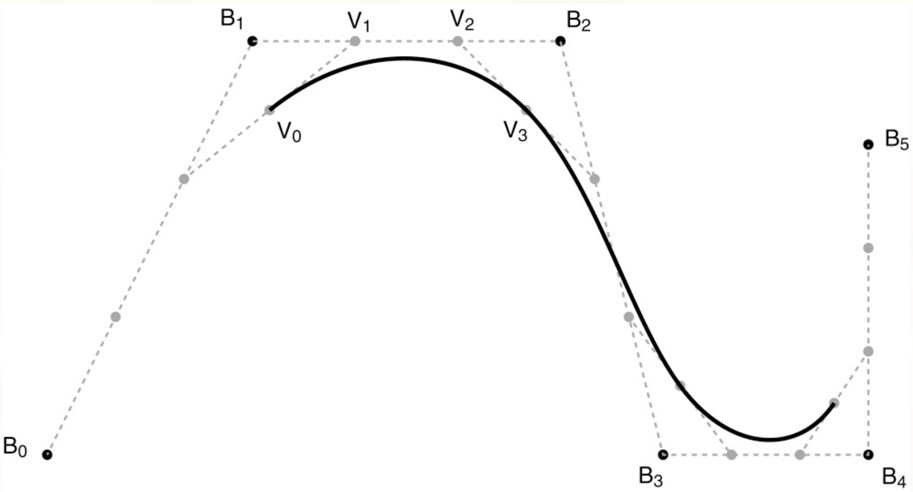
Tips for B-splines

B-spline:

- C2 continuity
- does not interpolate any ctrl points

Endpoints of B-splines

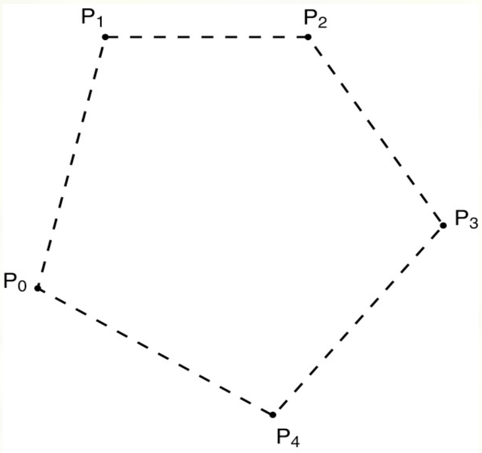
- We can see that B-splines don't interpolate the control points.
- It would be nice if we could at least control the *endpoints* of the splines explicitly.
- There's a trick to make the spline begin and end at control points by repeating them.
- In the example below, let's force interpolation of the last endpoint: (use endpoint 3 times)

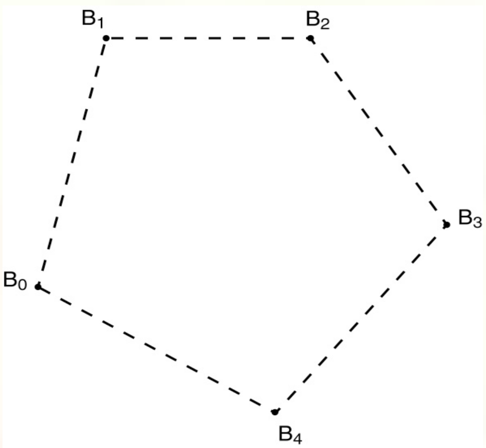


Tips for animator project

Closing the loop

- What if we want a closed curve, i.e., a loop?
- With Catmull-Rom and B-spline curves, this is easy:

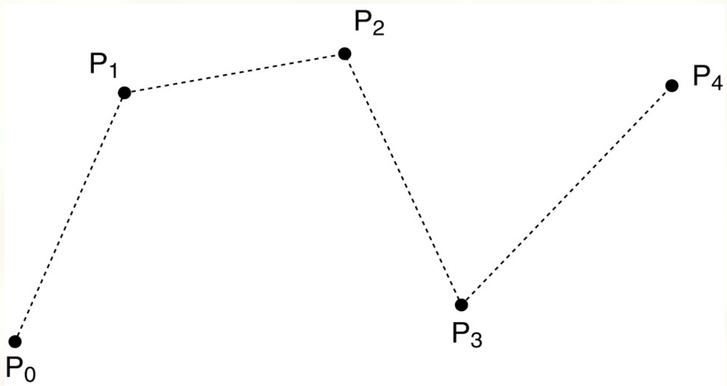




C2 interpolating curves

Simple interpolating splines

- Join several Hermite curves:
  - Make derivatives match
  - You still have ability to pick what that matched derivative is.

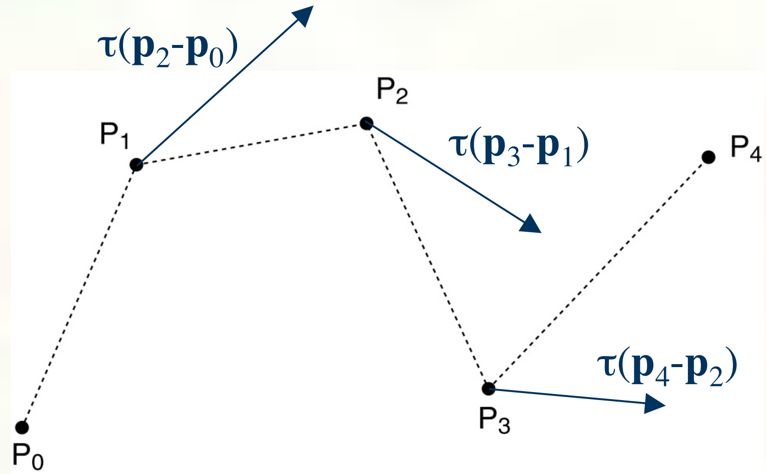


Cardinal splines

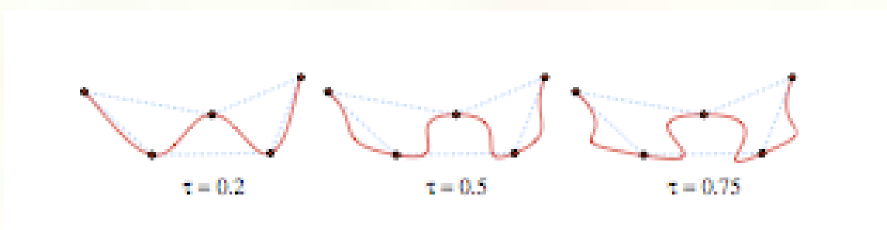
- If we set each derivative to be some positive scalar multiple *k* of the vector between the previous and next controls, we get a **Cardinal spline**.
- This leads to:

$$\mathbf{p}_i^u = \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1})$$

$$\mathbf{p}_{i+1}^u = \tau(\mathbf{p}_{i+2} - \mathbf{p}_i)$$



for any two consecutive interior points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ (we can deal with the endpoints separately if need be)



- Think of **τ** as a parameter that controls the **tension** of the spline

Cardinal splines

$$\mathbf{p}(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_i^u \\ \mathbf{p}_{i+1}^u \end{bmatrix}$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \tau(\mathbf{p}_{i+1} - \mathbf{p}_{i-1}) \\ \tau(\mathbf{p}_{i+2} - \mathbf{p}_i) \end{bmatrix}$$

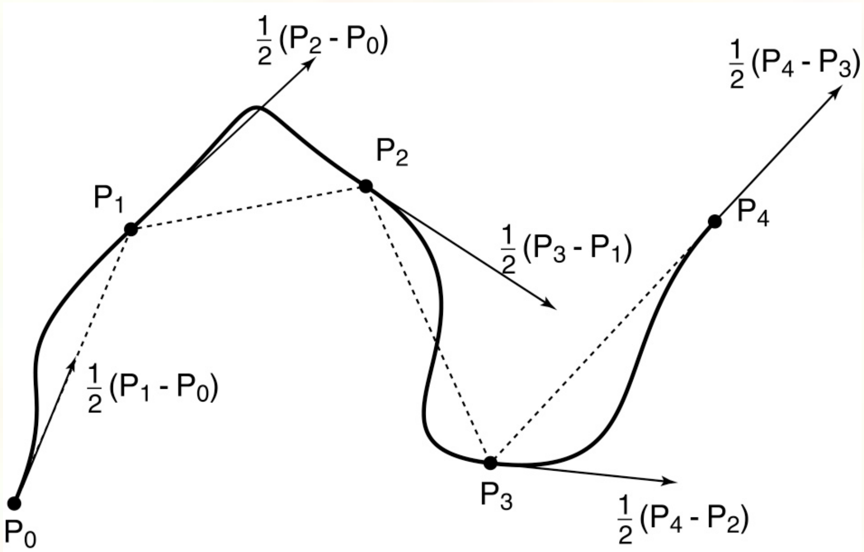
$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\tau & 0 & \tau & 0 \\ 0 & -\tau & 0 & \tau \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix}$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \tau \begin{bmatrix} -1 & 2/\tau - 1 & -2/\tau + 1 & 1 \\ 2 & -3/\tau + 1 & 3/\tau - 2 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1/\tau & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix}$$

Catmull-Rom splines

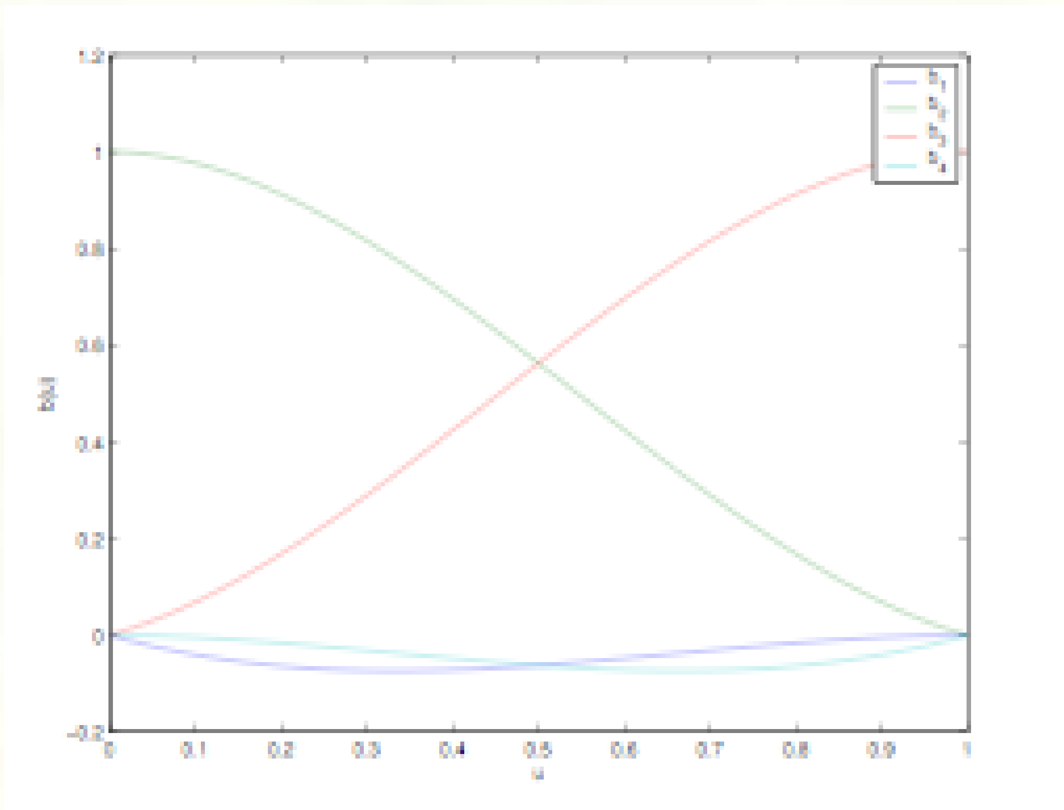
- If we set $\tau = 1/2$, we get a **Catmull-Rom spline**.
- So:

$$\mathbf{p}(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \tau \begin{bmatrix} -1 & 2/\tau - 1 & -2/\tau + 1 & 1 \\ 2 & -3/\tau + 1 & 3/\tau - 2 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1/\tau & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix}$$



for any two consecutive interior points $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ (again dealing with endpoints separately as needed)

Catmull-Rom blending functions



C^2 interpolating splines

- How can we keep the C^2 continuity we get with B-splines but get interpolation, too?
- Again start with connected cubic curves.
- Each cubic segment is an Hermite curve for which we get to set the position and derivative of the endpoints.
- That leaves us with a spline that's C^0 and C^1 such as a Catmull-Rom or Cardinal spline.
- But interestingly, there are other ways to choose the values of the (shared) first derivatives at the join points.
- Is there a way to set those derivatives to get other useful properties?

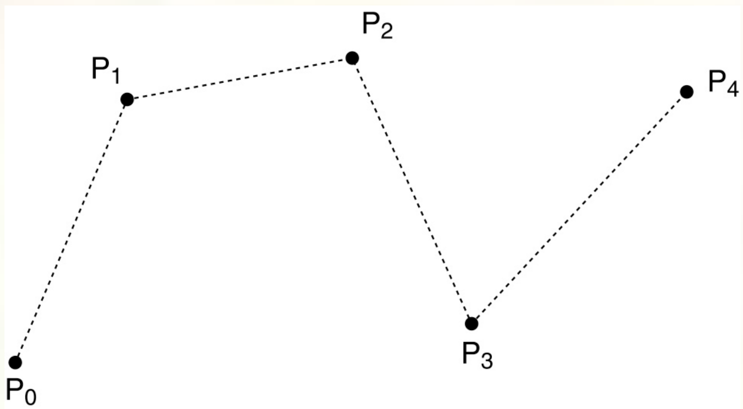
Find second derivatives

So far, we have:

C^0, C^1 continuity

Derivatives are still free, as '$D_0 \ldots D_4$'

Compute second derivatives at both sides of every join point:

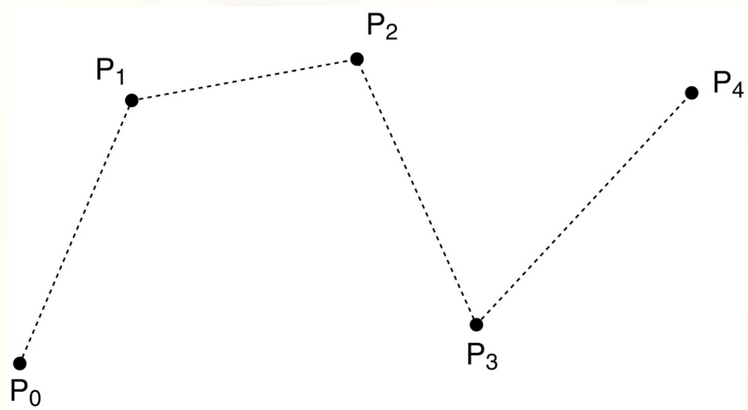


For $\mathbf{p}_1$: $Q_0''(1) = 6\mathbf{p}_0 - 6\mathbf{p}_1 + 2\mathbf{D}_0 + 4\mathbf{D}_1 \quad Q_1''(0) = -6\mathbf{p}_1 + 6\mathbf{p}_2 - 4\mathbf{D}_1 - 2\mathbf{D}_2$

For $\mathbf{p}_2$: $Q_1''(1) = 6\mathbf{p}_1 - 6\mathbf{p}_2 + 2\mathbf{D}_1 + 4\mathbf{D}_2 \quad Q_2''(0) = -6\mathbf{p}_2 + 6\mathbf{p}_3 - 4\mathbf{D}_2 - 2\mathbf{D}_3$

…

Match the second derivatives



Now, symbolically set the second derivatives to be equal.

For $\mathbf{p}_1$

$$6\mathbf{p}_0 - 6\mathbf{p}_1 + 2\mathbf{D}_0 + 4\mathbf{D}_1 = -6\mathbf{p}_1 + 6\mathbf{p}_2 - 4\mathbf{D}_1 - 2\mathbf{D}_2$$

$$3(\mathbf{p}_2 - \mathbf{p}_0) = \mathbf{D}_0 + 4\mathbf{D}_1 + \mathbf{D}_2$$

For $\mathbf{p}_2$

$$6\mathbf{p}_1 - 6\mathbf{p}_2 + 2\mathbf{D}_1 + 4\mathbf{D}_2 = -6\mathbf{p}_2 + 6\mathbf{p}_3 - 4\mathbf{D}_2 - 2\mathbf{D}_3$$

$$3(\mathbf{p}_3 - \mathbf{p}_1) = \mathbf{D}_1 + 4\mathbf{D}_2 + \mathbf{D}_3$$

…

Not quite done yet

- How many equations is this? m-1
- How many unknowns are we solving for? m+1
- We have two additional degrees of freedom, which we can nail down by imposing more conditions on the curve.
- There are various ways to do this. We'll use the variant called **natural C^2 interpolating splines**, which requires the second derivative to be zero at the endpoints.
- This condition gives us the two additional equations we need.
  - At the P_0 endpoint, it is: $Q_0''(0) = 0$
  - At the P_m endpoint, we have: $Q_{m-1}''(1) = 0$

Solving for the derivatives

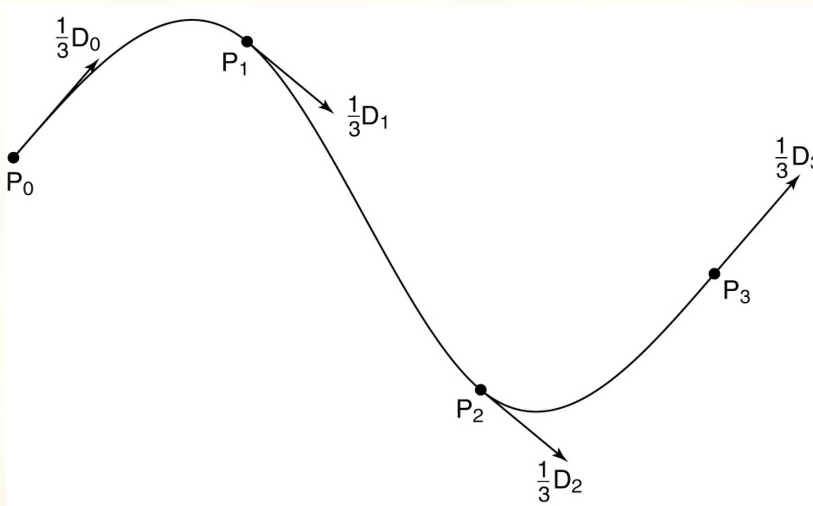
- Let's collect our m+1 equations into a single linear system:

$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{D}_0^T \\ \mathbf{D}_1^T \\ \mathbf{D}_2^T \\ \vdots \\ \mathbf{D}_{m-1}^T \\ \mathbf{D}_m^T \end{bmatrix} = \begin{bmatrix} 3(\mathbf{p}_1 - \mathbf{p}_0)^T \\ 3(\mathbf{p}_2 - \mathbf{p}_0)^T \\ 3(\mathbf{p}_3 - \mathbf{p}_1)^T \\ \vdots \\ 3(\mathbf{p}_m - \mathbf{p}_{m-2})^T \\ 3(\mathbf{p}_m - \mathbf{p}_{m-1})^T \end{bmatrix}$$

- It's easier to solve than it looks.
- See the notes from Bartels, Beatty, and Barsky for details.

C^2 interpolating spline

Once we've solved for the real $\mathbf{D}_i$s, we can plug them in to find our Bézier or Hermite curves and draw the final spline:



Have we lost anything?

=> Yes, local control.

Next time: Subdivision curves

- Basic idea:

  Represent a curve as an iterative algorithm, rather than as an explicit function.

- Reading:

  - Stollnitz, DeRose, and Salesin. Wavelets for Computer Graphics: Theory and Applications, 1996, section 6.1-6.3, A.5.
    [Course reader pp. 248-259 and pp. 273-274]