CS 395T
Lecture 7: Two-View Geometry

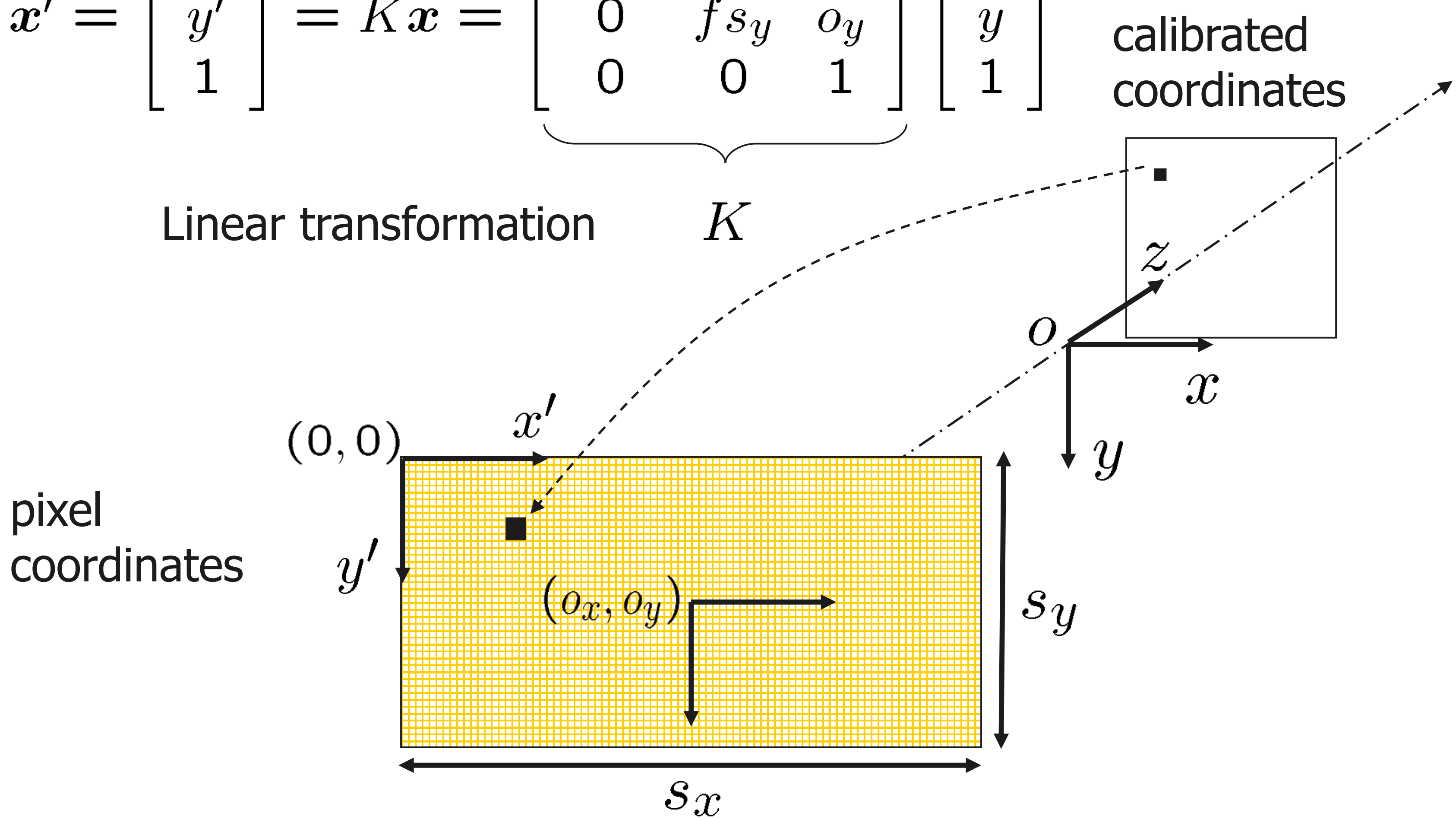
Qixing Huang

September 24th 2018

Uncalibrated Camera – Intrinsic Parameters are unknown

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K\boldsymbol{x} = \underbrace{\begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Linear transformation



Overview

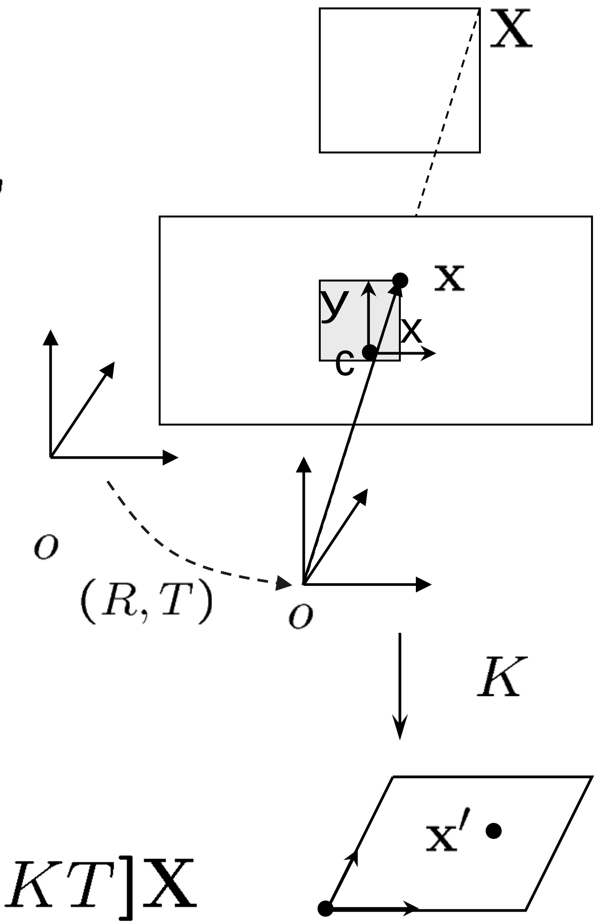
- Calibration with a rig (Checkborad for example)
-
- Uncalibrated epipolar geometry
-
- Ambiguities in image formation
-
- Stratified reconstruction

Uncalibrated Camera Using Homogeneous Coordinates

$$\mathbf{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1)$$

Last Lecture:

- Image plane coordinates $\mathbf{x} = [x, y, 1]^T$
- Camera extrinsic parameters $g = (R, T)$
- Perspective projection $\lambda \mathbf{x} = [R, T]\mathbf{X}$



This Lecture:

- Pixel coordinates $\mathbf{x}' = K\mathbf{x}$
-
- Projection matrix $\lambda \mathbf{x}' = \Pi \mathbf{X} = [KR, KT]\mathbf{X}$

Calibration with a Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated object (e.g., a cube) are known.

Calibration with a Rig

- Given 3-D coordinates on known object $\mathbf{X}$

$$\lambda \mathbf{x}' = [KR, KT]\mathbf{X} \quad \Rightarrow \quad \lambda \mathbf{x}' = \Pi \mathbf{X}$$

$$\lambda \begin{bmatrix} x^i \\ y^i \\ 1 \end{bmatrix} = \begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} \begin{bmatrix} X^i \\ Y^i \\ Z^i \\ 1 \end{bmatrix}$$

- Eliminate unknown scales

$$\begin{aligned} x^i(\pi_3^T \mathbf{X}) &= \pi_1^T \mathbf{X}, \\ y^i(\pi_3^T \mathbf{X}) &= \pi_2^T \mathbf{X} \end{aligned}$$

Calibration with a Rig

- Recover projection matrix $\Pi = [KR, KT] = [R', T']$

$$\Pi^s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]^T$$

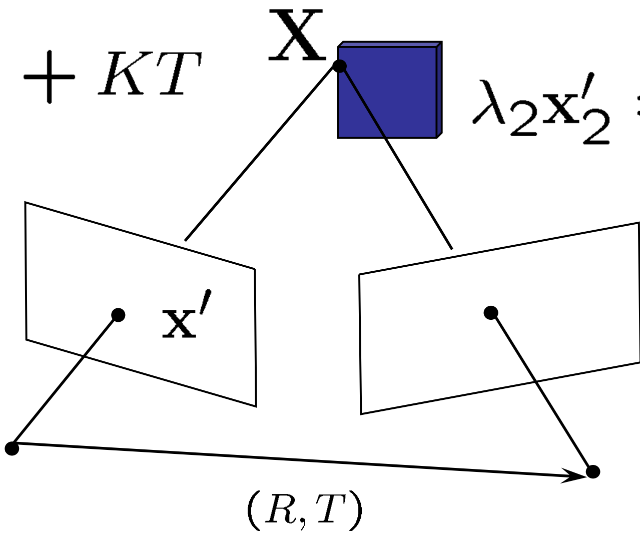
$$\min \|M\Pi^s\|^2 \quad \text{subject to} \quad \|\Pi^s\|^2 = 1$$

Again singular value decomposition

- Factor the KR into $R \in SO(3)$ and K using QR decomposition
- Solve for translation $T = K^{-1}T'$

Uncalibrated Epipolar Geometry

$$\lambda_2 K \mathbf{x}_2 = K R \lambda_1 \mathbf{x}_1 + K T$$



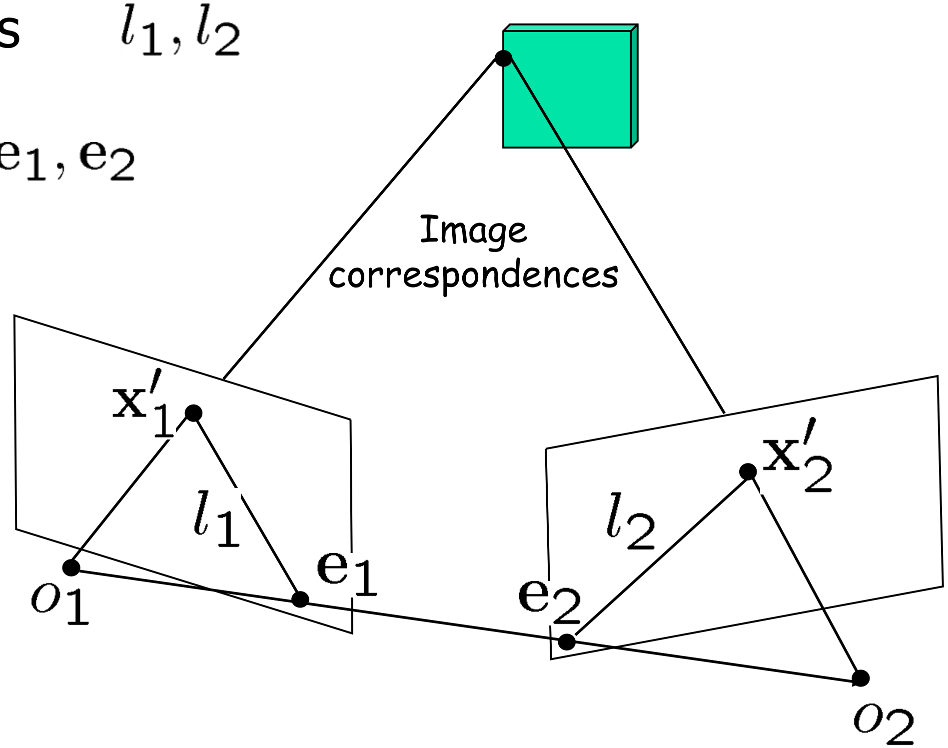
$$\lambda_2 \mathbf{x}'_2 = K R K^{-1} \lambda_1 \mathbf{x}'_1 + T'$$

- Epipolar constraint $\mathbf{x}'^T_2 \underbrace{K^{-T} \hat{T} R K^{-1}} \mathbf{x}'_1 = 0$
-
- Fundamental matrix $F = K^{-T} \hat{T} R K^{-1}$
-
- Equivalent forms of $F = K^{-T} \hat{T} R K^{-1} = \hat{T}' K R K^{-1}$

Properties of the Fundamental Matrix

$$\mathbf{x'}_2^T F \mathbf{x'}_1 = 0$$

- Epipolar lines l_1, l_2
-
- Epipoles $\mathbf{e}_1, \mathbf{e}_2$



$$l_1 \sim F^T \mathbf{x}'_2 \qquad l_i^T \mathbf{x}'_i = 0 \qquad l_2 \sim F \mathbf{x}'_1$$

$$F \mathbf{e}_1 = 0 \qquad l_i^T \mathbf{e}_i = 0 \qquad \mathbf{e}_2^T F = 0$$

Properties of the Fundamental Matrix

Remark 6.1. *Characterization of the fundamental matrix. A non-zero matrix* $F \in \mathbb{R}^{3\times 3}$ *is a fundamental matrix if* F *has a singular value decomposition (SVD):* $E = U\Sigma V^T$ *with*

$$\Sigma = diag\{\sigma_1, \sigma_2, 0\}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}_+$.

There is little structure in the matrix F except that

$$\det(F) = 0$$

Estimating Fundamental Matrix

- Find such F that the epipolar error is minimized

$$\min_{F} \sum_{j=1}^{n} (\boldsymbol{x}'_{2,j}{}^{T} F \boldsymbol{x}'_{1,j})^2 \quad s.t. \quad \|F\|_{\mathcal{F}}^2 = 1$$

- Fundamental matrix can be estimated up to scale
- Denote $\mathbf{a} = \mathrm{x}'_1 \otimes \mathrm{x}'_2$

$\mathbf{a} = [x_1x_2, x_1y_2, x_1z_2, y_1x_2, y_1y_2, y_1z_2, z_1x_2, z_1y_2, z_1z_2]^T$

$F^s = [f_1, f_4, f_7, f_2, f_5, f_8, f_3, f_6, f_9]^T$

- Rewrite $\mathrm{a}^T F^s = 0$

$$\min_{F^s} \|AF^s\|^2 \quad s.t. \quad \|F^s\|^2 = 1$$

Two view linear algorithm – 8-point algorithm

- Solve the LLSE problem:

$$\min_F \sum_{j=1}^{n} ({\boldsymbol{x}'_{2,j}}^T F \boldsymbol{x}'_{1,j})^2 \quad s.t. \quad \|F\|^2_{\mathcal{F}} = 1$$

- Solution eigenvector associated with smallest eigenvalue of A^TA

- Compute SVD of F recovered from data

$$F = U\Sigma V^T \quad \Sigma = diag(\sigma_1, \sigma_2, \sigma_3)$$

- Project onto the essential manifold:

$$\Sigma' = diag(\sigma_1, \sigma_2, 0) \quad F = U\Sigma' V^T$$

- F cannot be unambiguously decomposed into pose and calibration

$$F = K^{-T}\hat{T}RK^{-1}$$

What Does *F* Tell Us?

- *F* can be inferred from point matches (eight-point algorithm)
- Cannot extract motion, structure and calibration from one fundamental matrix (two views)
- *F* allows reconstruction up to a projective transformation (as we will see soon)
- *F* encodes all the geometric information among two views when no additional information is available

Decomposing the Fundamental Matrix

$$F = K^{-T}\widehat{T}RK^{-1} = \widehat{T}'KRK^{-1}$$

- Decomposition of the fundamental matrix into a skew-symmetric matrix and a nonsingular matrix

$$F \mapsto \Pi = [R', T'] \quad \Rightarrow \quad F = \widehat{T'}R'.$$

- Decomposition of F is not unique

$$\mathbf{x}_2'\widehat{T}'(T'v^T + KRK^{-1})\mathbf{x}_1' = 0 \quad T' = KT$$

- Unknown parameters - ambiguity

$$v = [v_1, v_2, v_3]^T \in \Re^3, \quad v_4 \in \Re$$

- Corresponding projection matrix

$$\Pi = [KRK^{-1} + T'v^T, v_4T']$$

Projective Reconstruction

- From image correspondences, extract F, followed by computation of projection matrices Π_{ip} and structure $\mathbf{X}_p$
- Canonical decomposition

$$F \quad \mapsto \quad \Pi_{1p} = [I,\ 0],\ \Pi_{2p} = [(\widehat{T'})^T F,\ T']$$

- Given projection matrices --- recover structure $\mathbf{X}_p$

$$\begin{aligned} \lambda_1 \mathbf{x}'_1 &= \Pi_{1p}\mathbf{X}_p &= [I, 0]\mathbf{X}_p, \\ \lambda_2 \mathbf{x}'_2 &= \Pi_{2p}\mathbf{X}_p &= [(\widehat{T'})^T F, T']\mathbf{X}_p. \end{aligned}$$

- Projective ambiguity --- non-singular 4x4 matrix

$$\begin{aligned} \lambda_i \mathbf{x}'_i &= \Pi_{ip} H^{-1} H \mathbf{X}_p \\ \lambda_i \mathbf{x}'_i &= \tilde{\Pi}_{1p}\tilde{\mathbf{X}}_p \end{aligned}$$

Both Π_{ip} and $\tilde{\Pi}_{ip}$ are consistent with the epipolar geometry – give the same fundamental matrix

Projective Reconstruction

- Given projection matrices recover projective structure

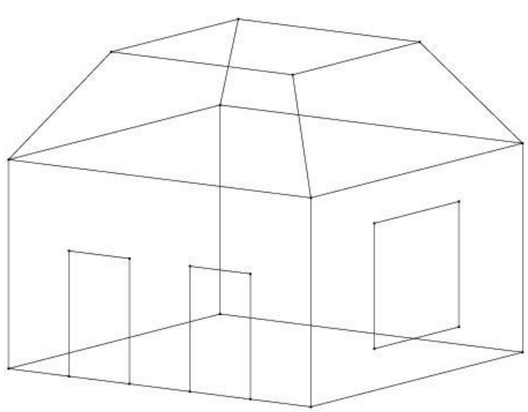
$$
\begin{aligned}
(x_1\pi_1^{3T})\mathbf{X}_p &= \pi_1^{1T}\mathbf{X}_p, & (y_1\pi_1^{3T})\mathbf{X}_p &= \pi_1^{2T}\mathbf{X}_p,\\
(x_2\pi_2^{3T})\mathbf{X}_p &= \pi_2^{1T}\mathbf{X}_p, & (y_2\pi_2^{3T})\mathbf{X}_p &= \pi_2^{2T}\mathbf{X}_p,
\end{aligned}
$$

- This is a linear problem and can be solve using linear least-squares

$$M\mathbf{X}_p = 0$$

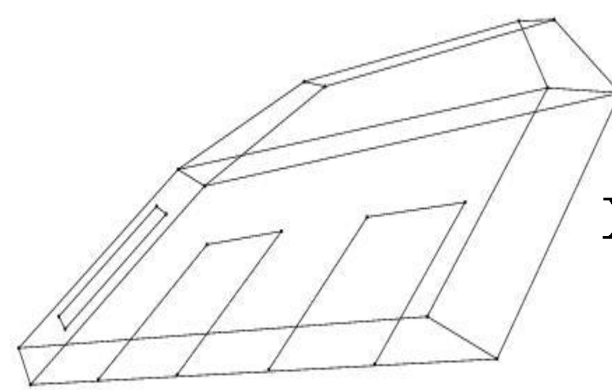
- Projective reconstruction – projective camera matrices and projective structure

$\mathbf{X}_e$



Euclidean Structure

$$\mathbf{X}_e = H\mathbf{X}_p$$



$\mathbf{X}_p$

Projective Structure

Euclidean vs Projective reconstruction

- Euclidean reconstruction – true metric properties of objects lenghts (distances), angles, parallelism are preserved
- Unchanged under rigid body transformations

- Projective reconstruction – lengths, angles, parallelism are NOT preserved – we get distorted images of objects – their distorted 3D counterparts --> 3D projective reconstruction

Ambiguities in Image Formation

- Structure of the (uncalibrated) projection matrix

$$\lambda \mathbf{x}' = \Pi \mathbf{X} = (\Pi H^{-1})(H\mathbf{X}) = \tilde{\Pi}\tilde{\mathbf{X}} \quad \Pi = [KR, KT]$$

- For any invertible 4x4 matrix H
- In the uncalibrated case we cannot distinguish between Π camera imaging word $\mathbf{X}$ from camera $\tilde{\Pi}$ imaging distorted world $\tilde{\mathbf{X}}$
- In general, H is of the following form

$$H^{-1} = \begin{bmatrix} G & b \\ v^T & v_4 \end{bmatrix}$$

- In order to preserve the choice of the first reference frame we can restrict some DOF of H

Structure of the Projective Ambiguity

- 1st frame as reference

$$\lambda_1 \mathbf{x}'_1 = K_1 \Pi_0 \mathbf{X}_e$$

$$\lambda_1 \mathbf{x}'_1 = K_1 \Pi_0 H^{-1} H \mathbf{X}_e = \Pi_{1p} \mathbf{X}_p$$

- Choose the projective reference frame

$\Pi_{1p} = [I_{3\times 3}, 0]$ then ambiguity is $H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix}$

H^{-1} can be further decomposed as

$$H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \doteq H_a^{-1} H_p^{-1}$$

Stratified (Euclidean) Reconstruction

- General ambiguity – while preserving choice of first reference frame

$$H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix}$$

- Decomposing the ambiguity into affine and projective one

$$H^{-1} = H_a^{-1} H_p^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}$$

- Note the different effect of the 4-th homogeneous coordinate

Will continue next lecture