CS354 Computer Graphics
Surface Representation III

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Today's Topic

- Bspline curve operations (Brief)
  - Knot Insertion/Deletion
- Subdivision (Focus)
  - Subdivision curves
  - Subdivision surfaces

Bspline

- Polynomials of order k that are stitched together with C^{k-1} continuity

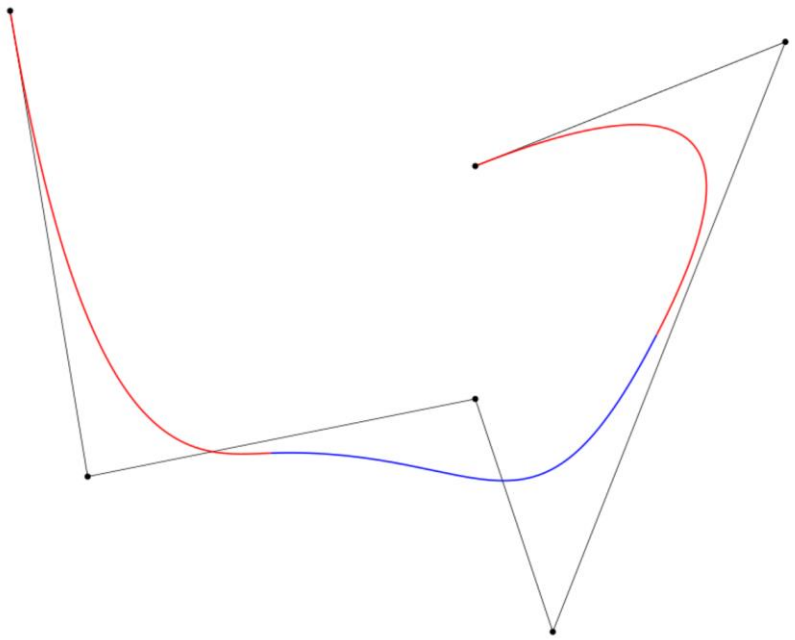


Image from https://en.wikipedia.org/wiki/B-spline

Knot insertion

- Break a curve segments into two segments
  - [t1, t3] to [t1, t2] and [t2, t3]
  - C^k continuity at knot t2
  - Becomes C^{k-1} continuity after moving the control point

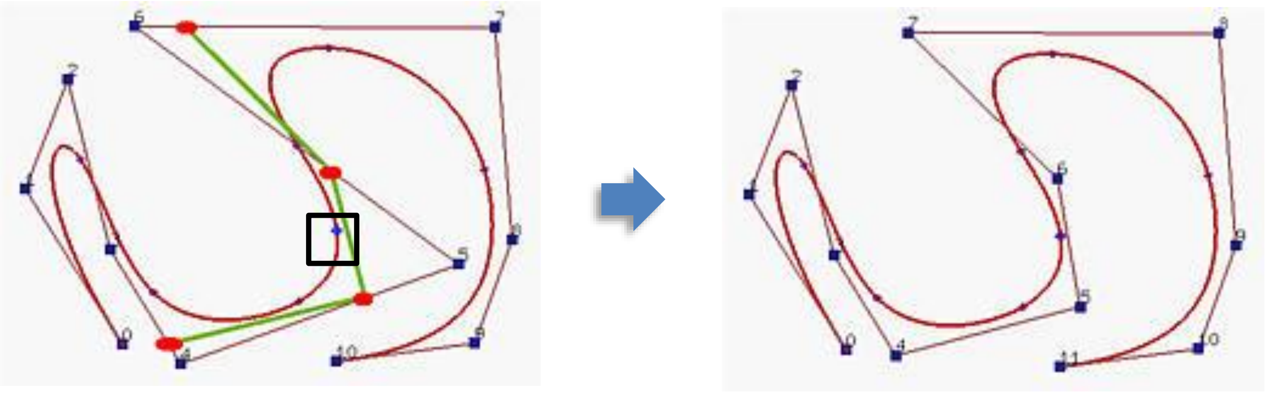
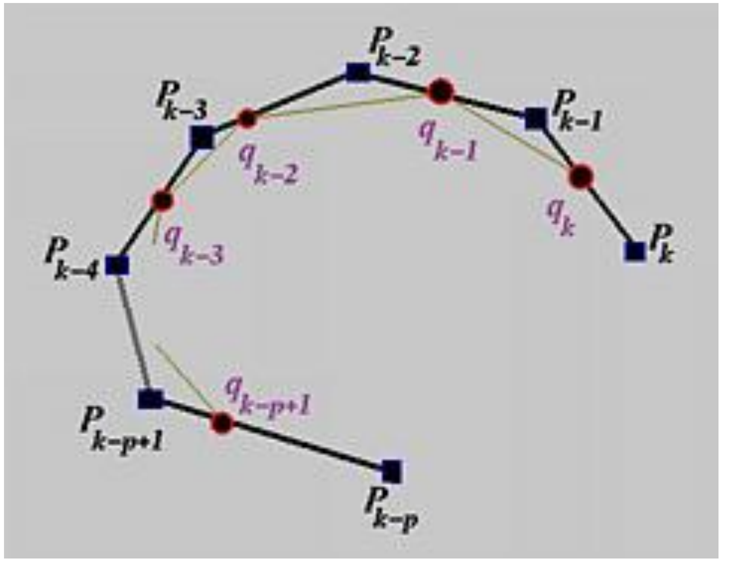


Image from http://pages.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/single-insertion.html

Knot insertion



$$\mathbf{Q}_i = (1 - a_i)\mathbf{P}_{i-1} + a_i\mathbf{P}_i \qquad a_i = \frac{t - u_i}{u_{i+p} - u_i} \qquad \text{for } k - p + 1 \leq i \leq k$$

Image from http://pages.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/single-insertion.html

Knot removal

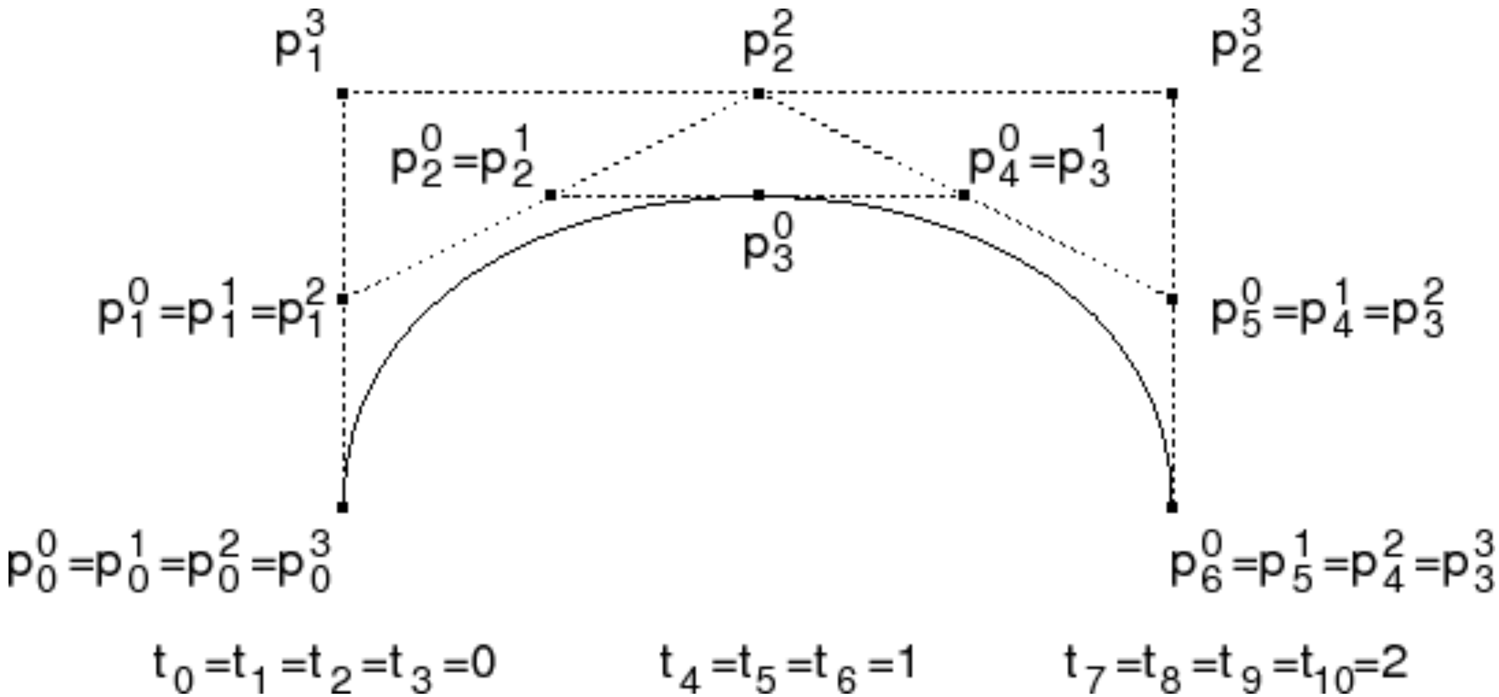


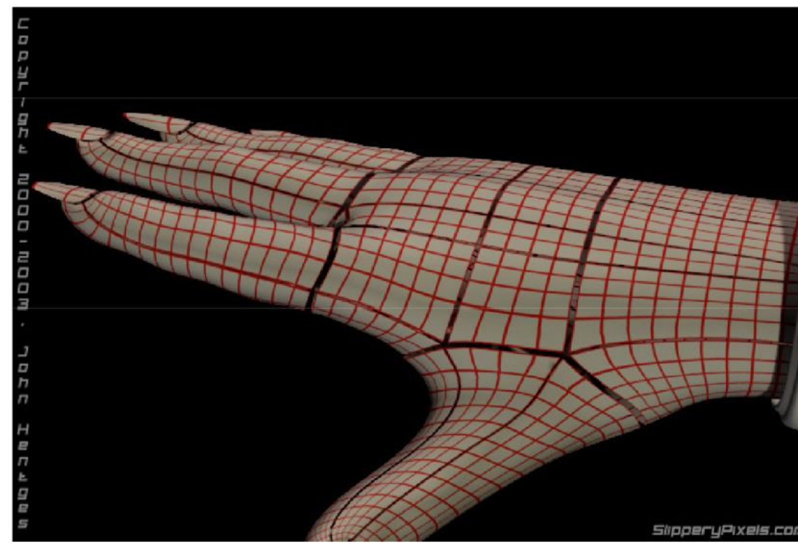
Image from http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node18.html

Subdivision Curves/Surfaces

Slide Credit: Mirela-Ben Chen

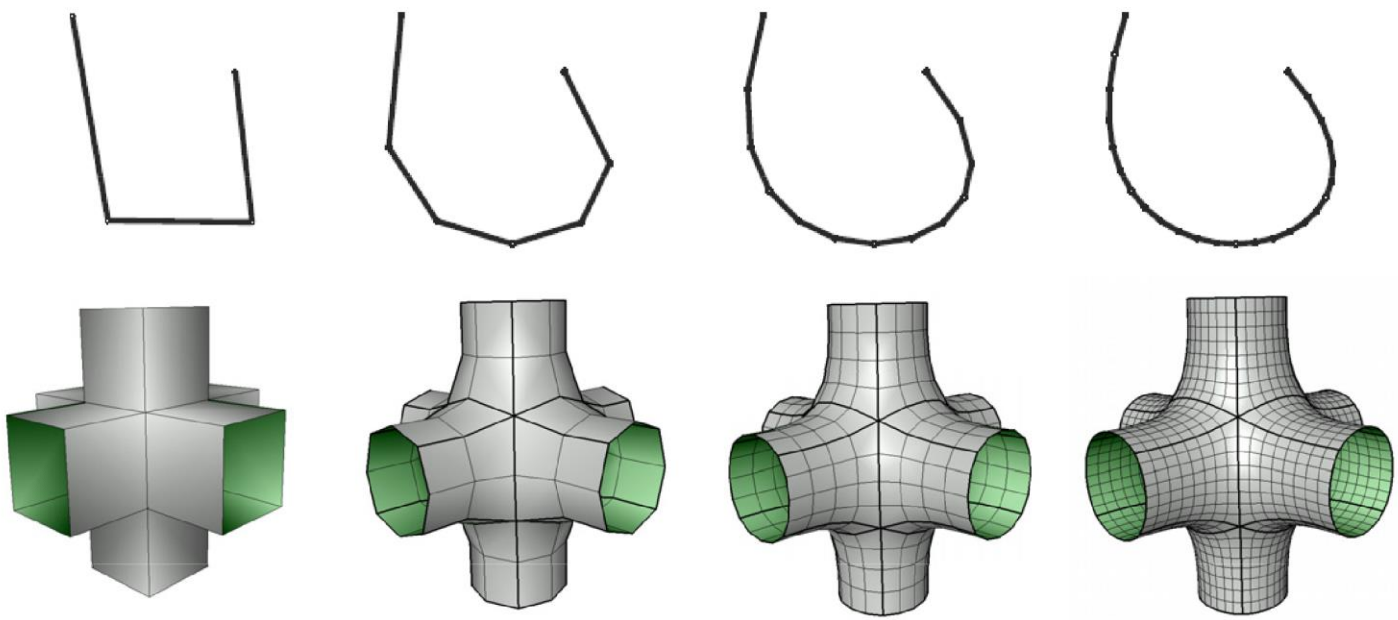
Problems with NURBS

- A single NURBS patch is either a disk a tube or a torus
- Must use many NURBS patches to model complex geometry
- When deforming a surface made of NURBS patches,cracks arise at the seams



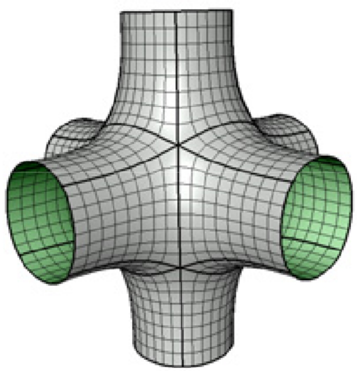
Subdivision

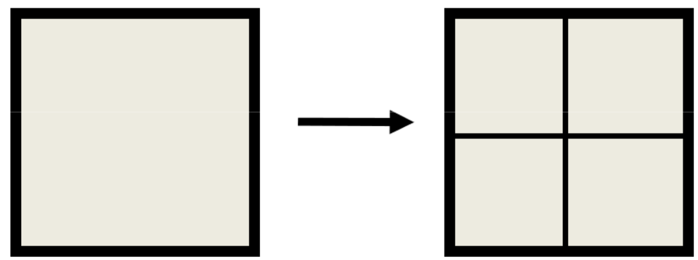
- “Subdivision defines a smooth curve or surfac eaes the limit of a sequence of successive refi nements”



Subdivision Rules

- How the connectivity changes

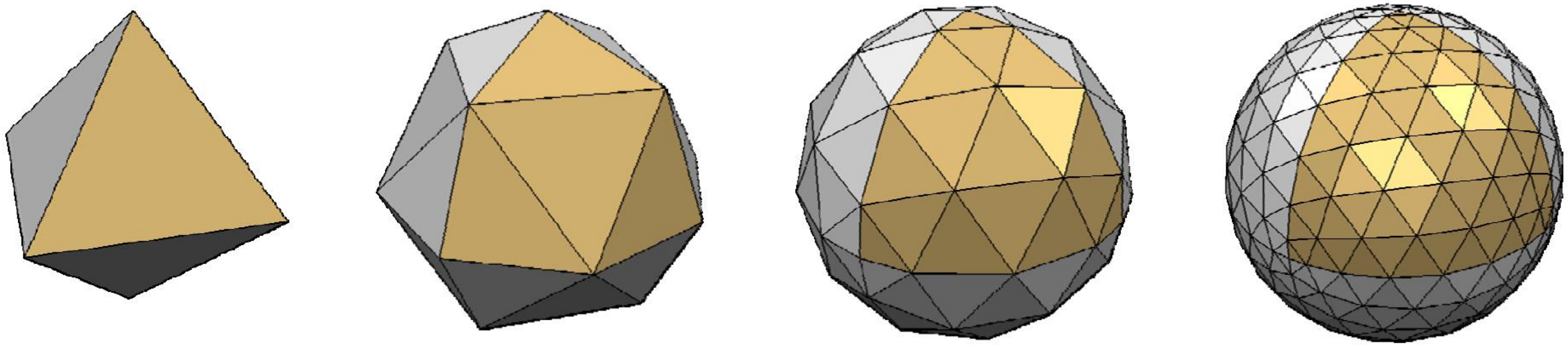




- How the geometry changes
  - Old points
  - New points

Subdivision Surfaces

- Generalization of spline curves / surfaces
  - Arbitrary control meshes
  - Successive refinement (subdivision)
  - Converges to smooth limit surface
  - Connection between splines and meshes



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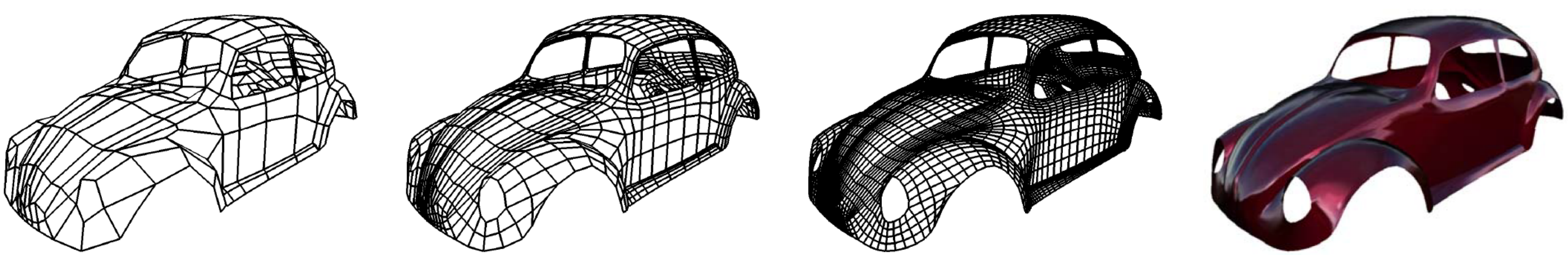
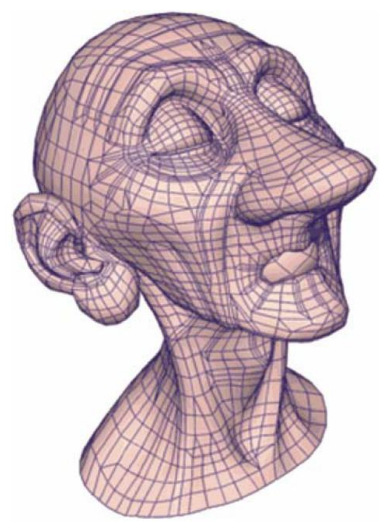


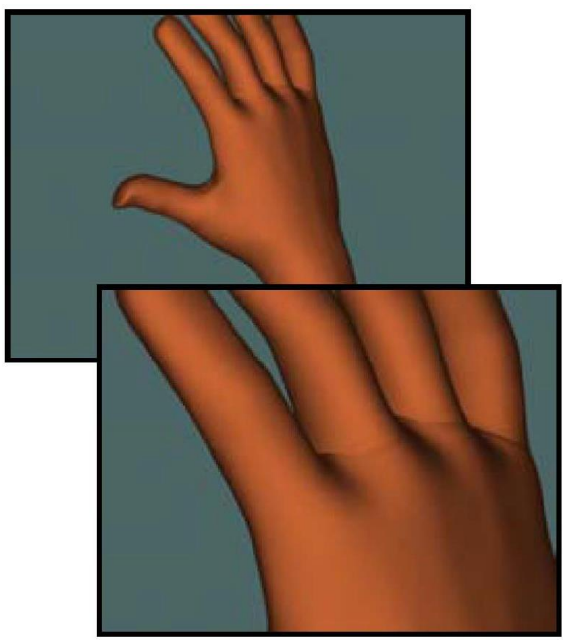
Example: Geri's Game (Pixar)

- Subdivision used for
  - Geri's hands and head
  - Clothing
  - Tie and shoes

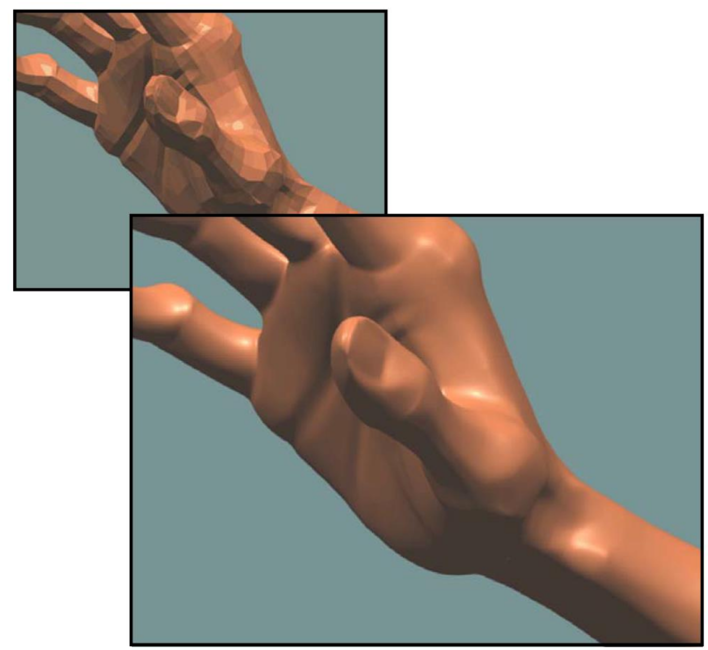


Example: Geri’s Game (Pixar)

Woody’s hand (NURBS)



Geri’s hand (subdivision)

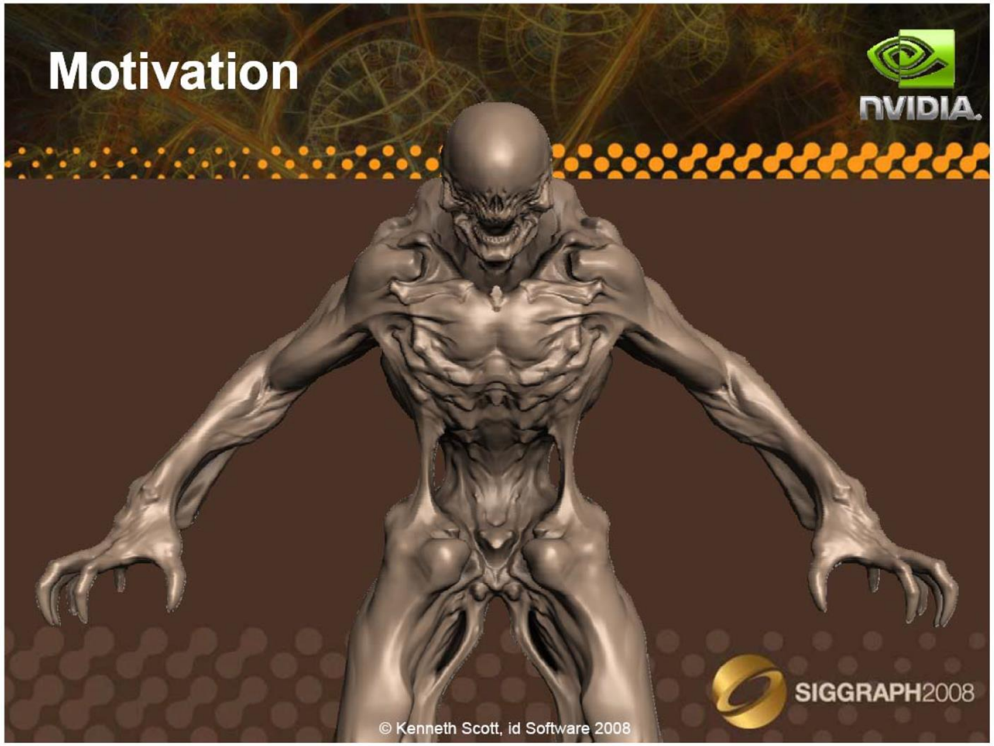


Example: Geri’s Game (Pixar)

Sharp and semi-sharp features

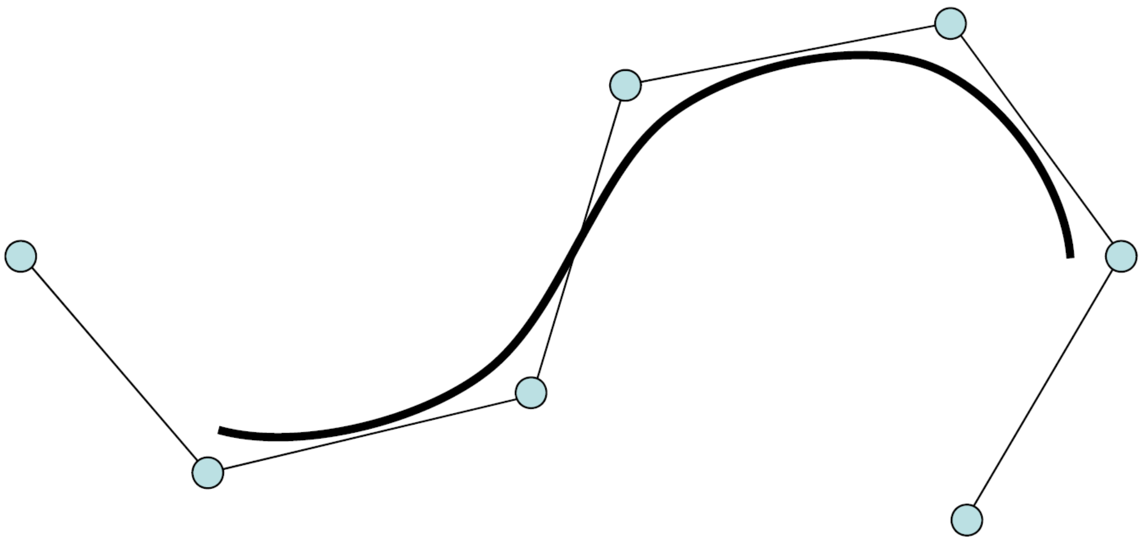
Example: Games

- Supported in hardware in DirectX 11



Subdivision curves

Given a control polygon...



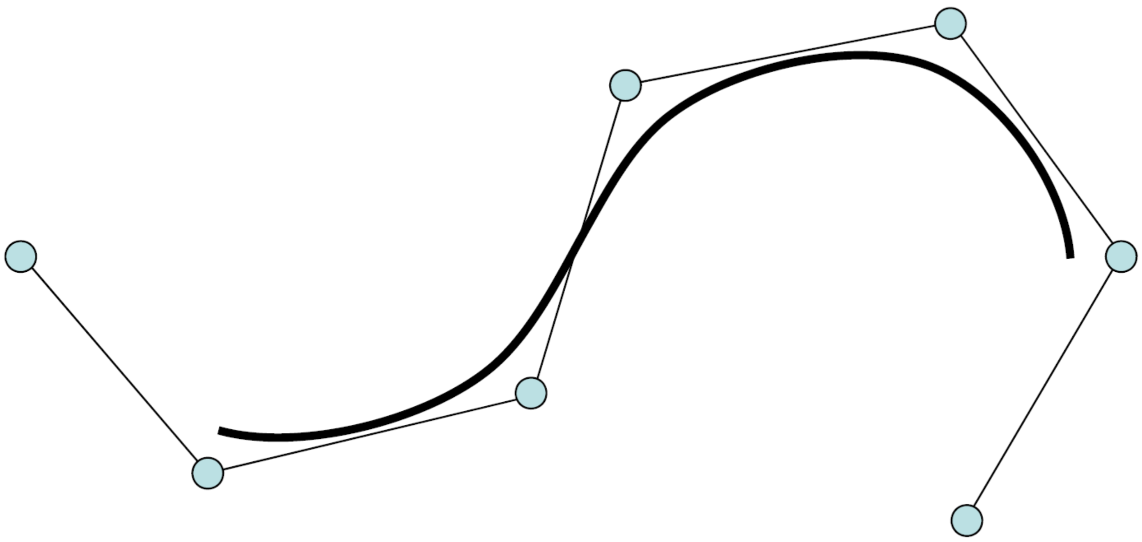
...find a smooth curve related to that polygon

Subdivision curve types

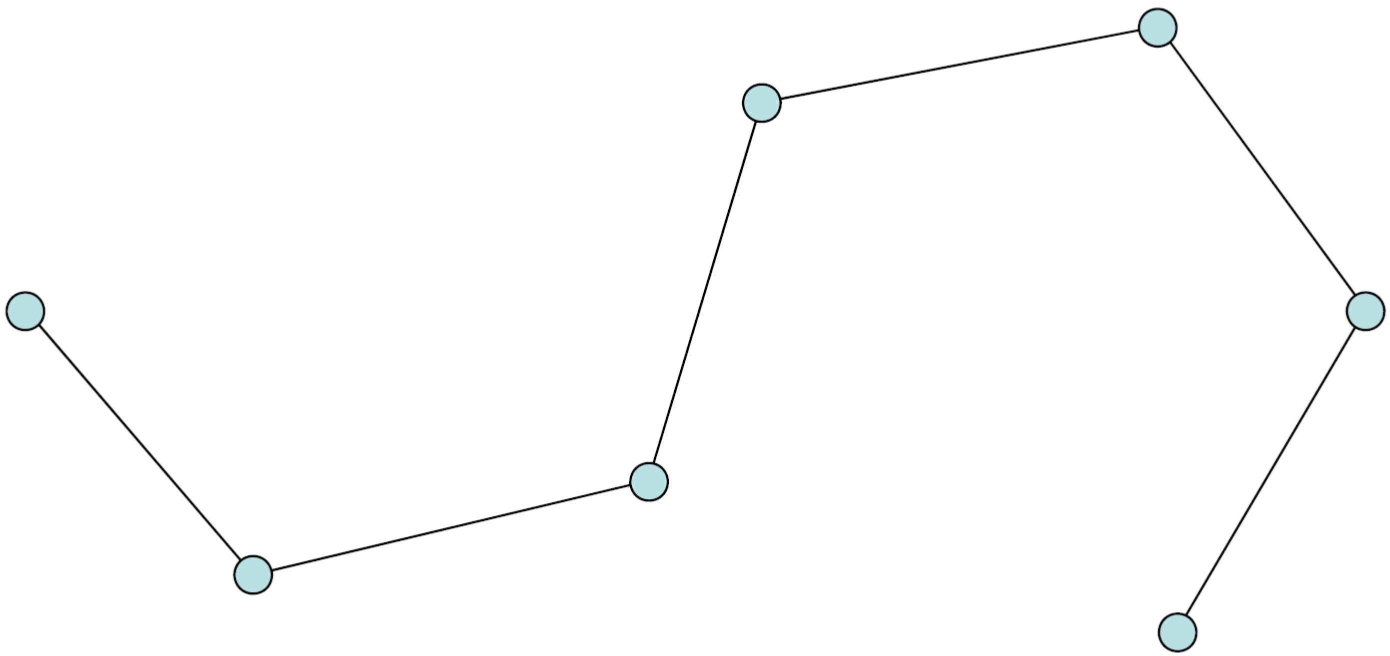
Approximating

Interpolating

Corner Cutting

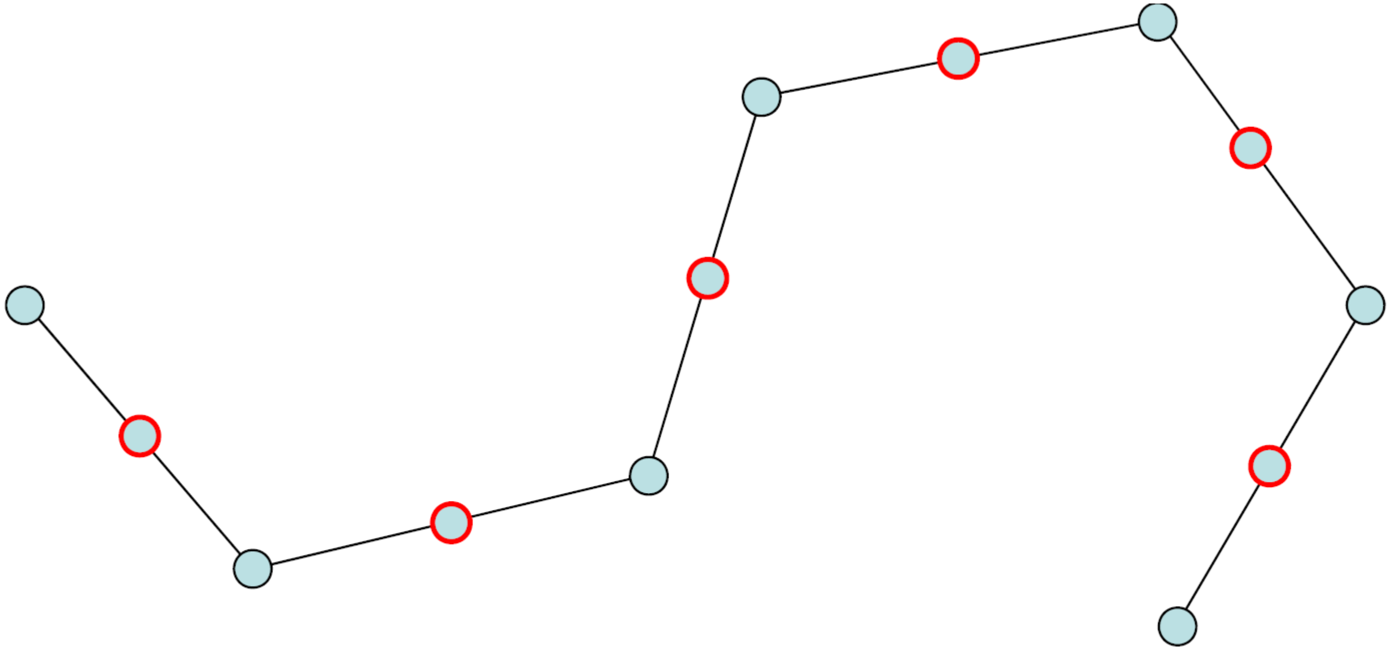


Approximating



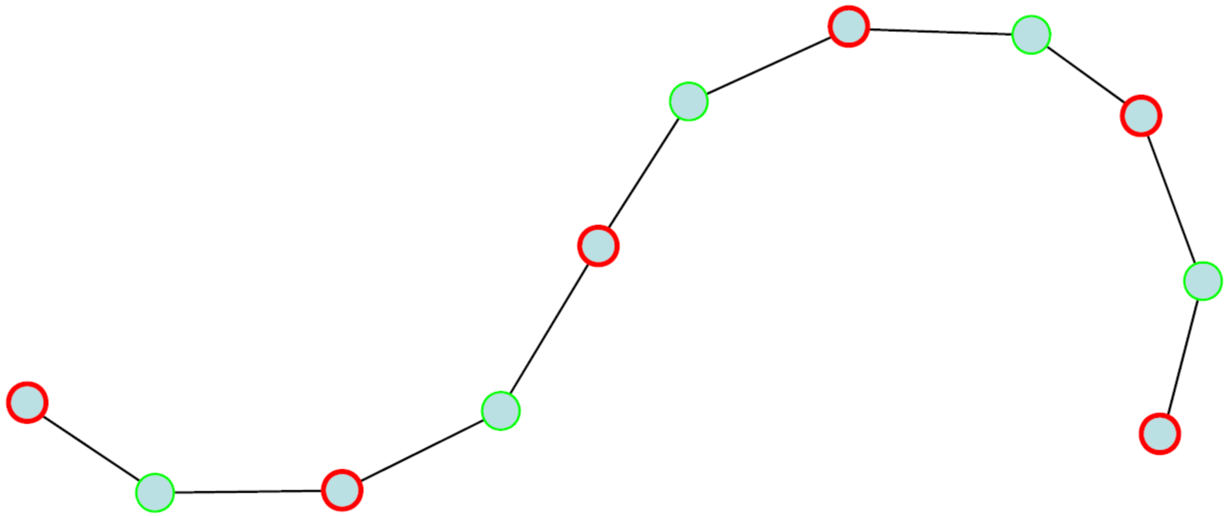
Approximating

Splitting step: split each edge in two



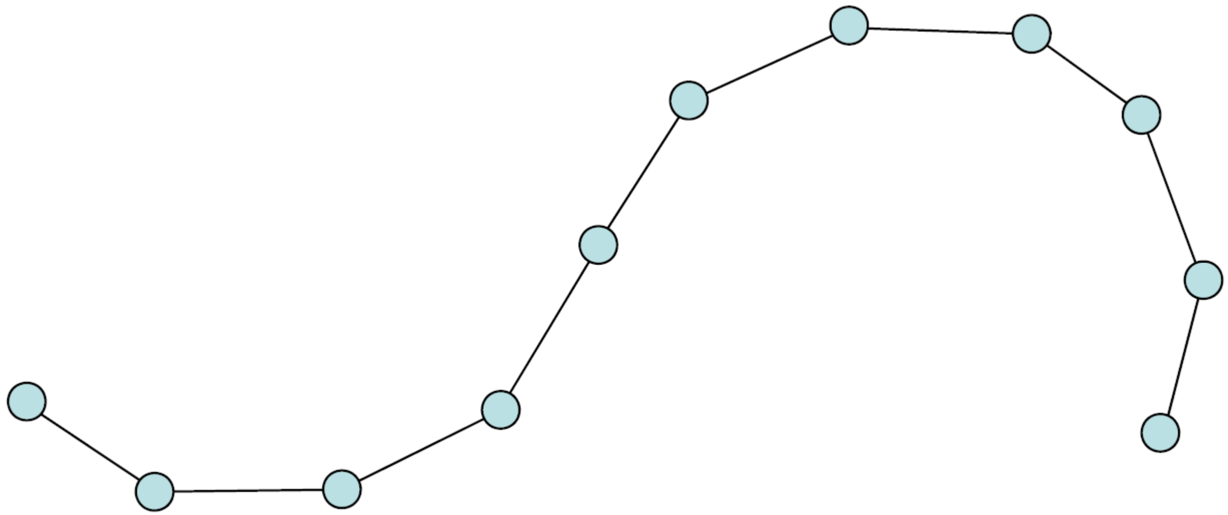
Approximating

Averaging step: relocate each (original) vertex according to some (simple) rule...



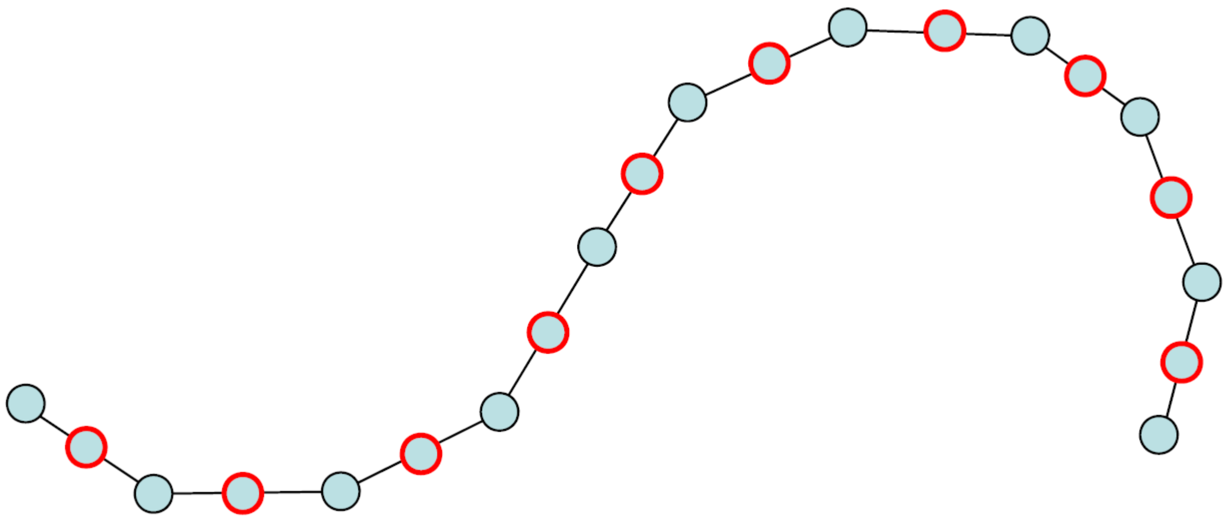
Approximating

Start over ...



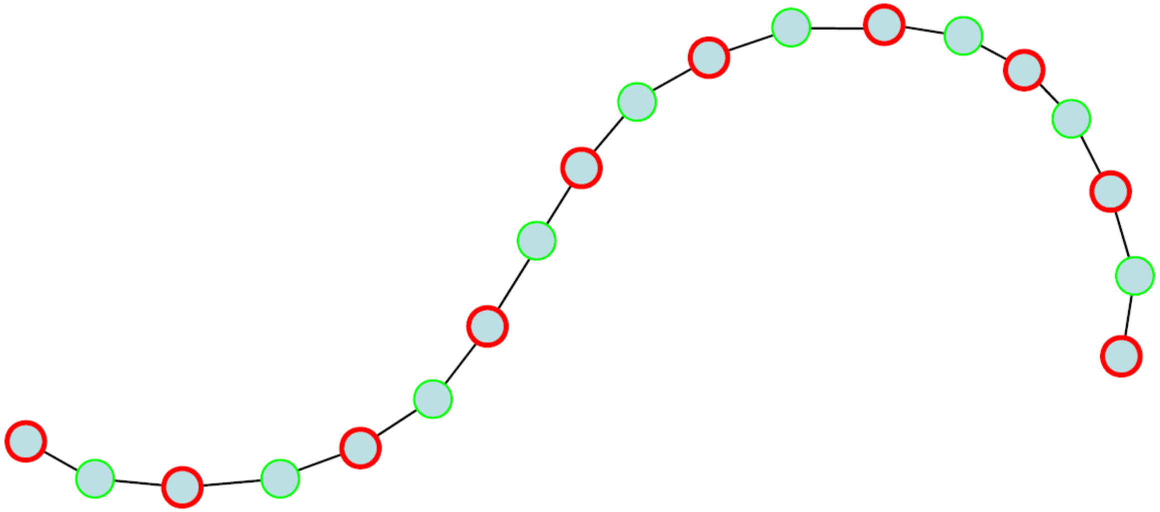
Approximating

...splitting...



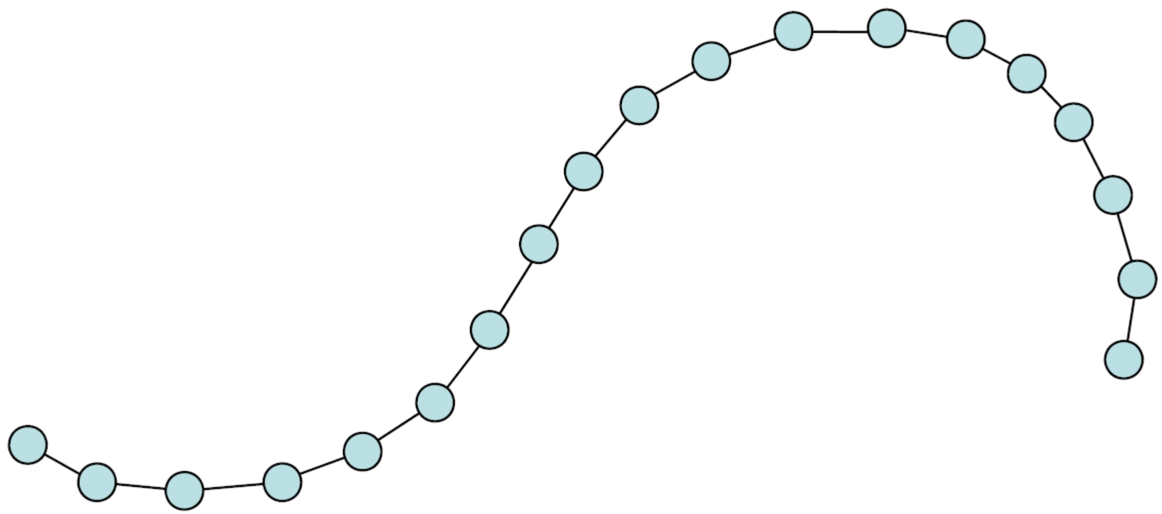
Approximating

...averaging...



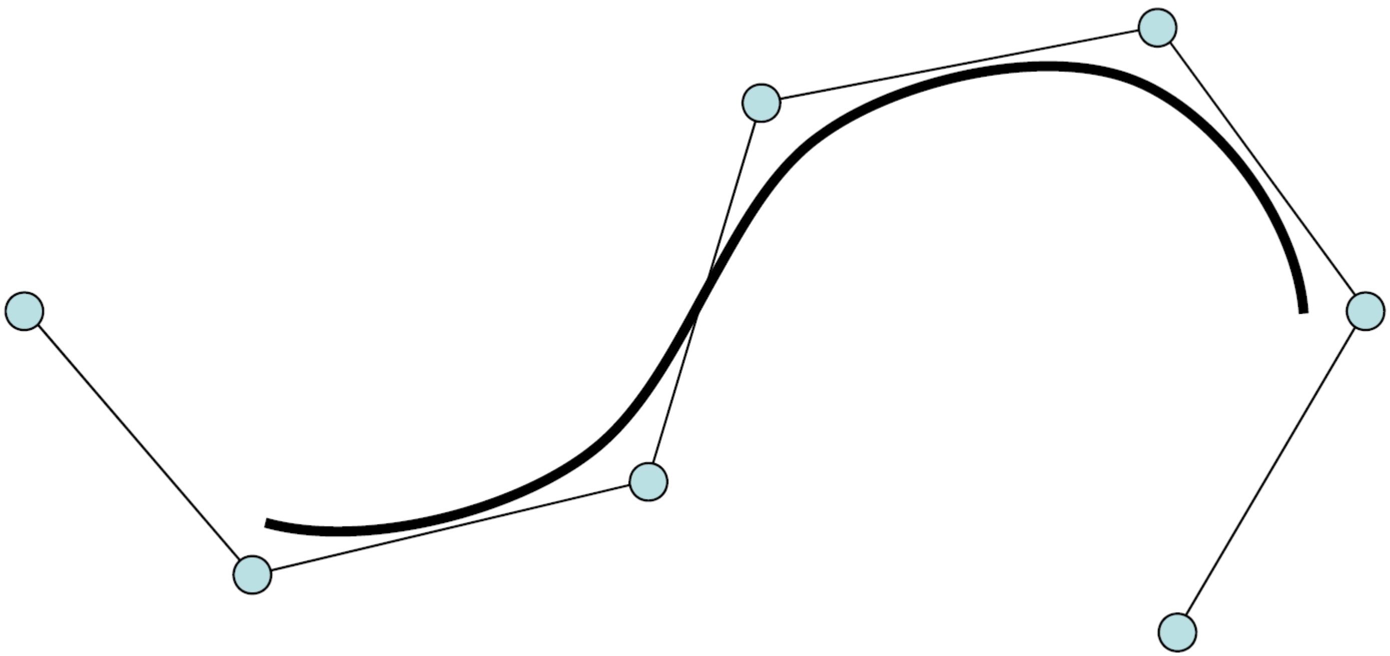
Approximating

...and so on...



Approximating

If the rule is designed carefully...



... the control polygons will converge to a smooth limit curve!

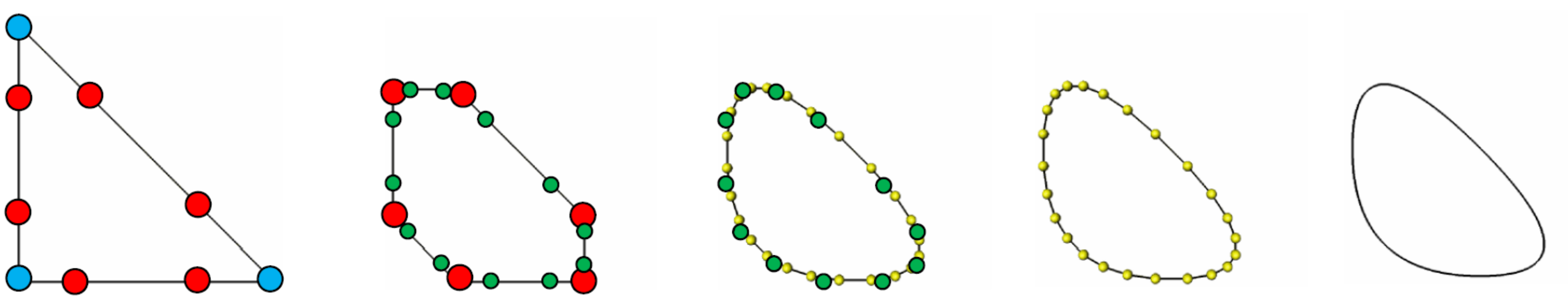
Equivalent to …

- Insert *single* new point at mid-edge
- *Filter* entire set of points

Catmull-Clark rule: Filter with (1/8, 6/8, 1/8)

Corner Cutting

- Subdivision rule:
  - Insert *two* new vertices at ¼ and ¾ of each edge
  - *Remove* the old vertices
  - Connect the new vertices



B-spline curves

- Piecewise polynomial of degree *n*

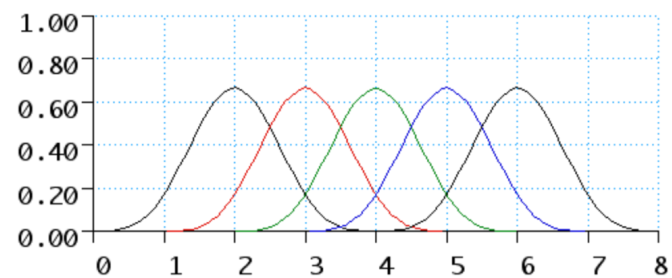
B-spline curve

control points

$$\mathbf{s}(u) = \sum_{i=0}^{k} \mathbf{d}_i N_i^n(u)$$

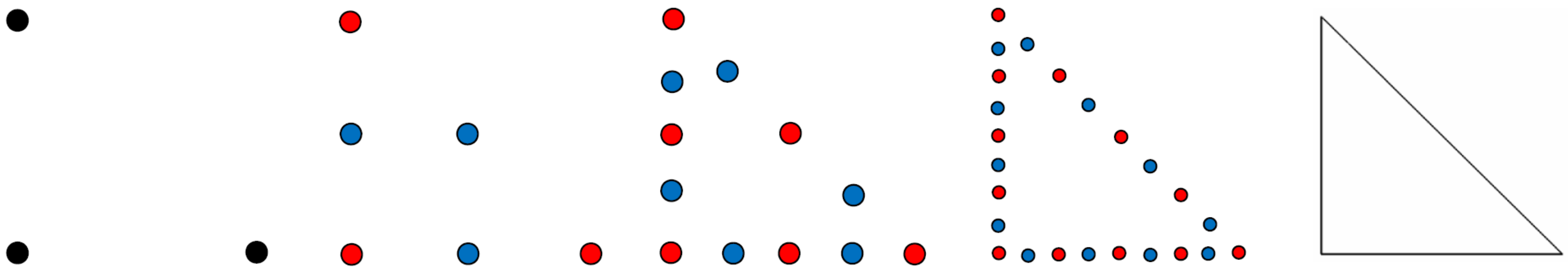
parameter value

basis functions



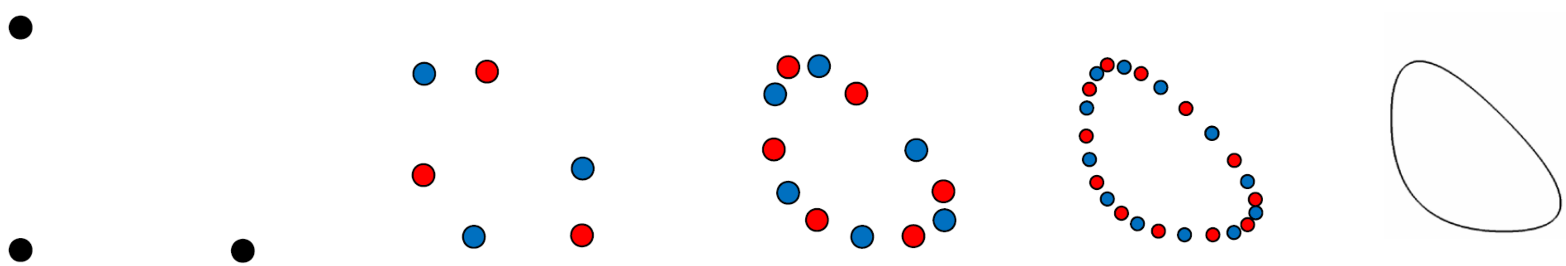
B-spline curves

- Distinguish between odd and even points
- **Linear B-spline**
  - Odd coefficients (1/2, 1/2)
  - Even coefficient (1)



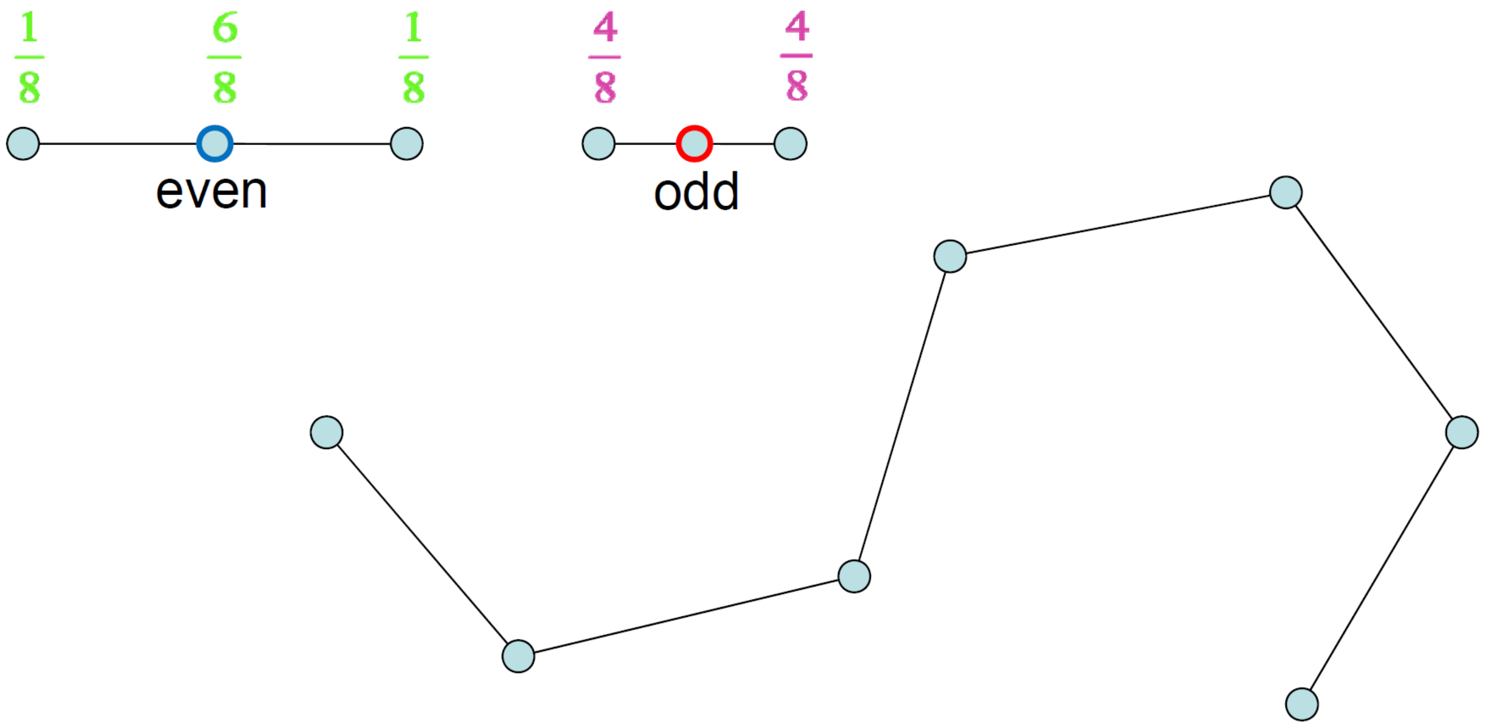
B-spline curves

- **Quadratic B Spline (Chaikin)**
  - Odd coefficients (¼, ¾)
  - Even coefficients (¾ , ¼)

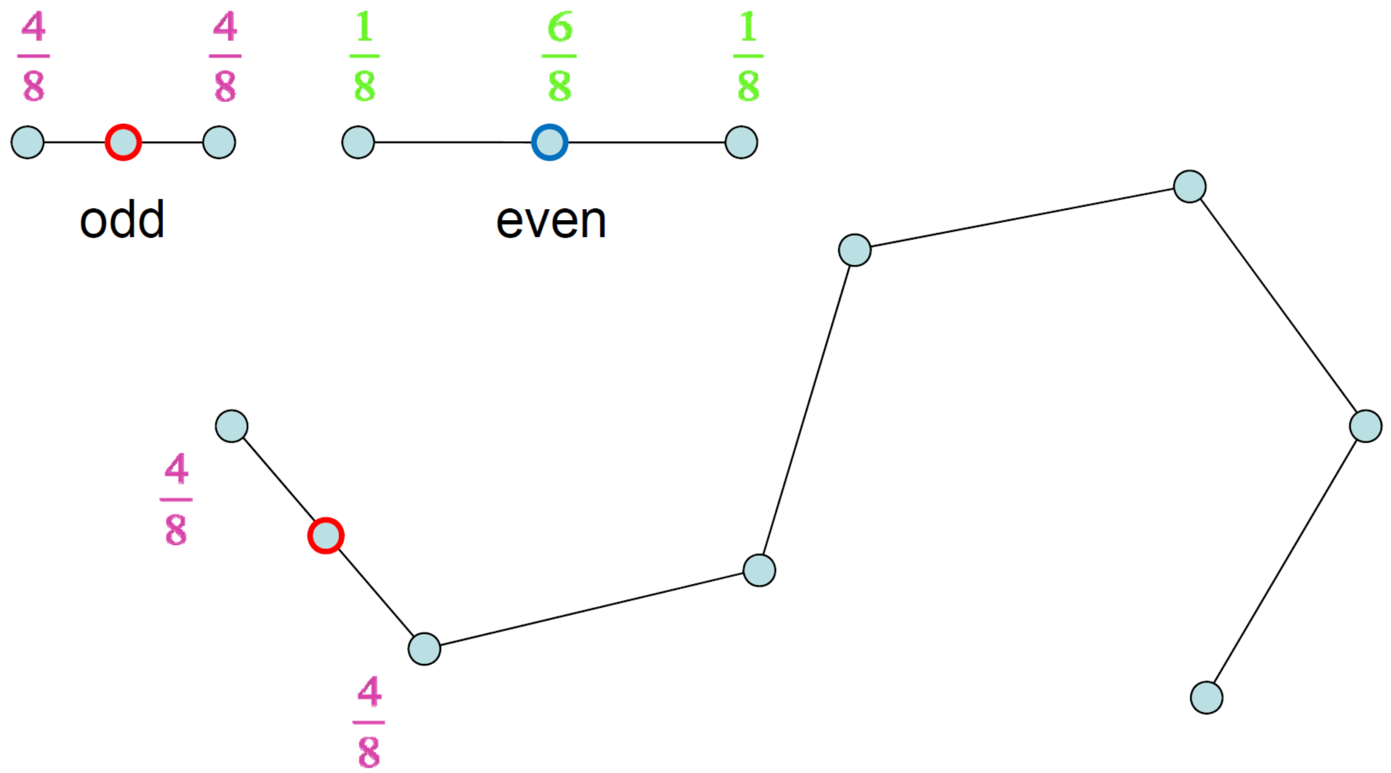


- **Cubic B-Spline (Catmull-Clark)**
  - Odd coefficients (4/8, 4/8)
  - Even coefficients (1/8, 6/8, 1/8)

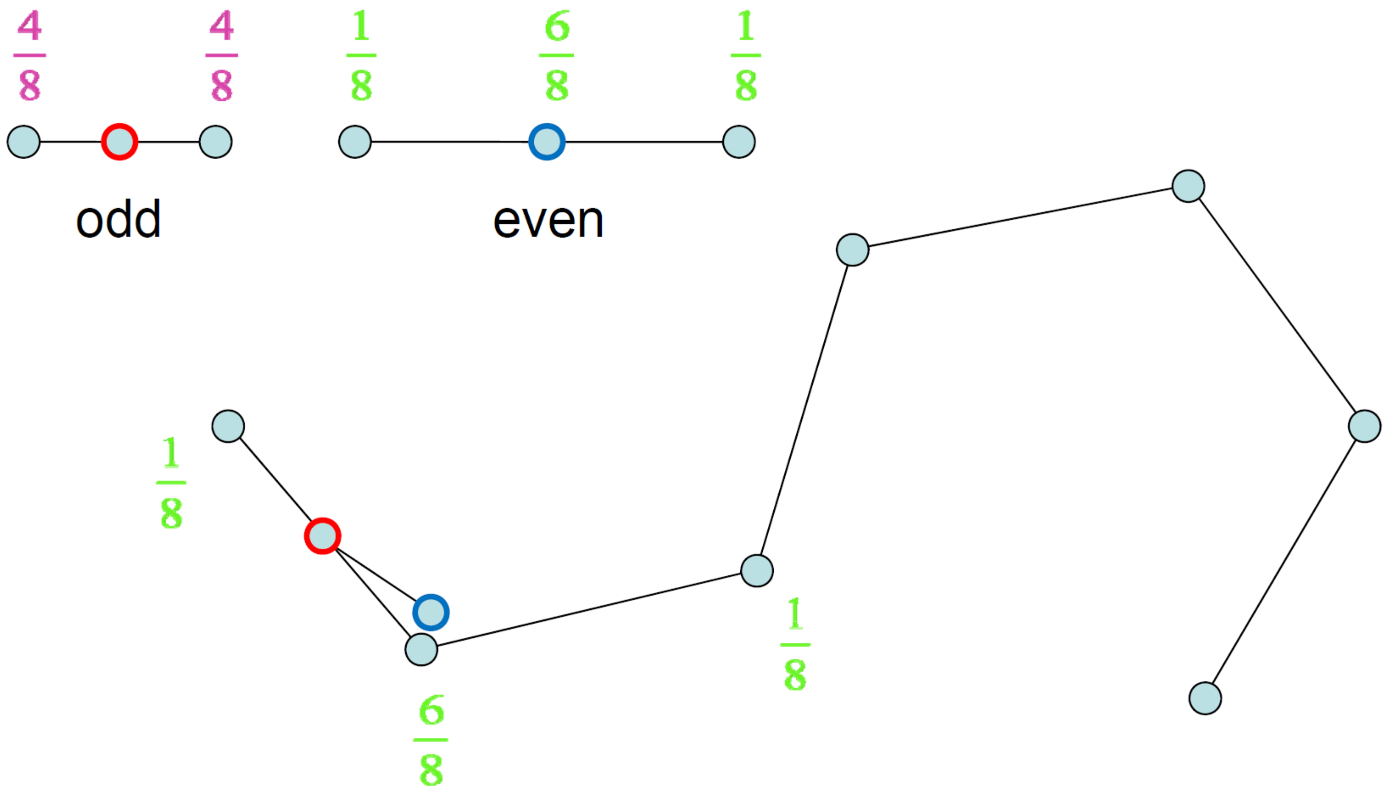
Cubic B-Spline



Cubic B-Spline



Cubic B-Spline



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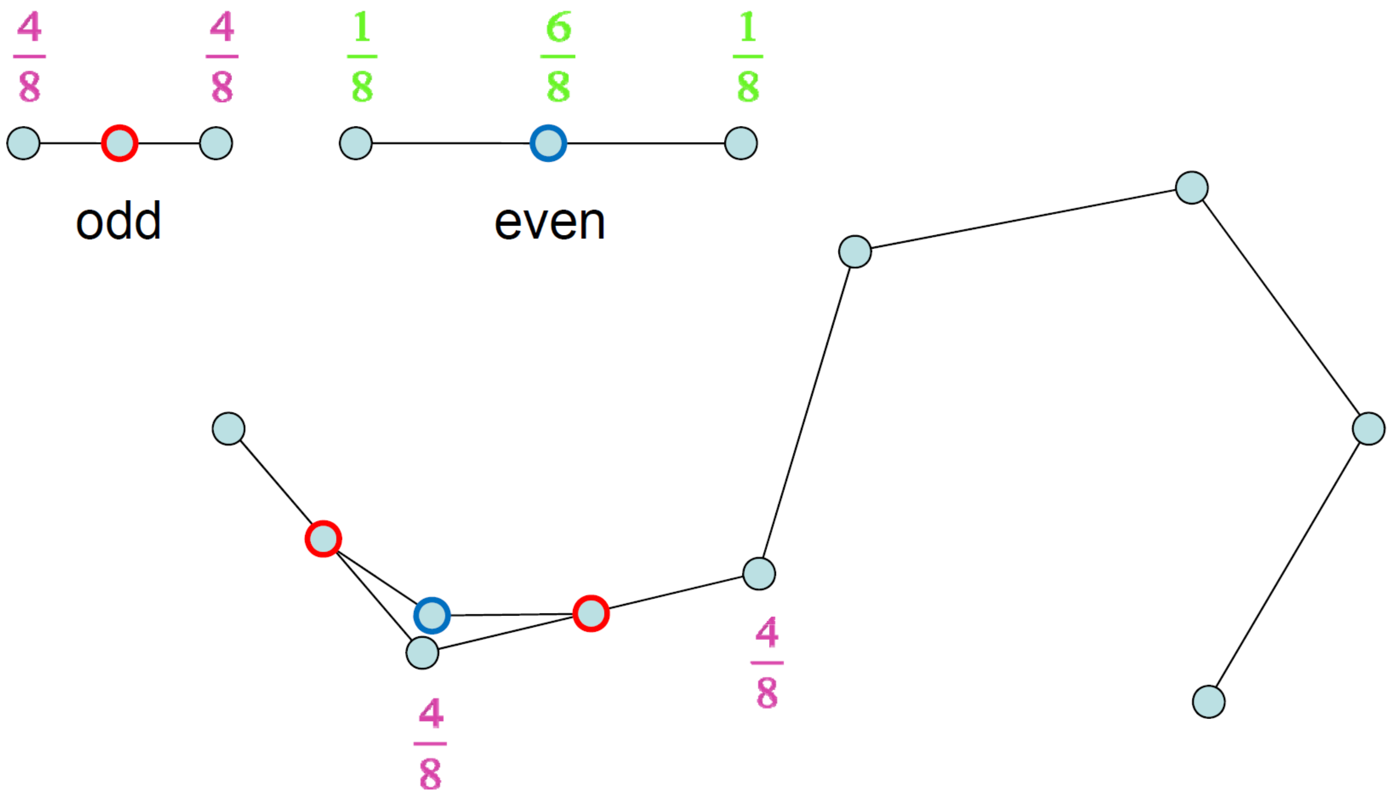

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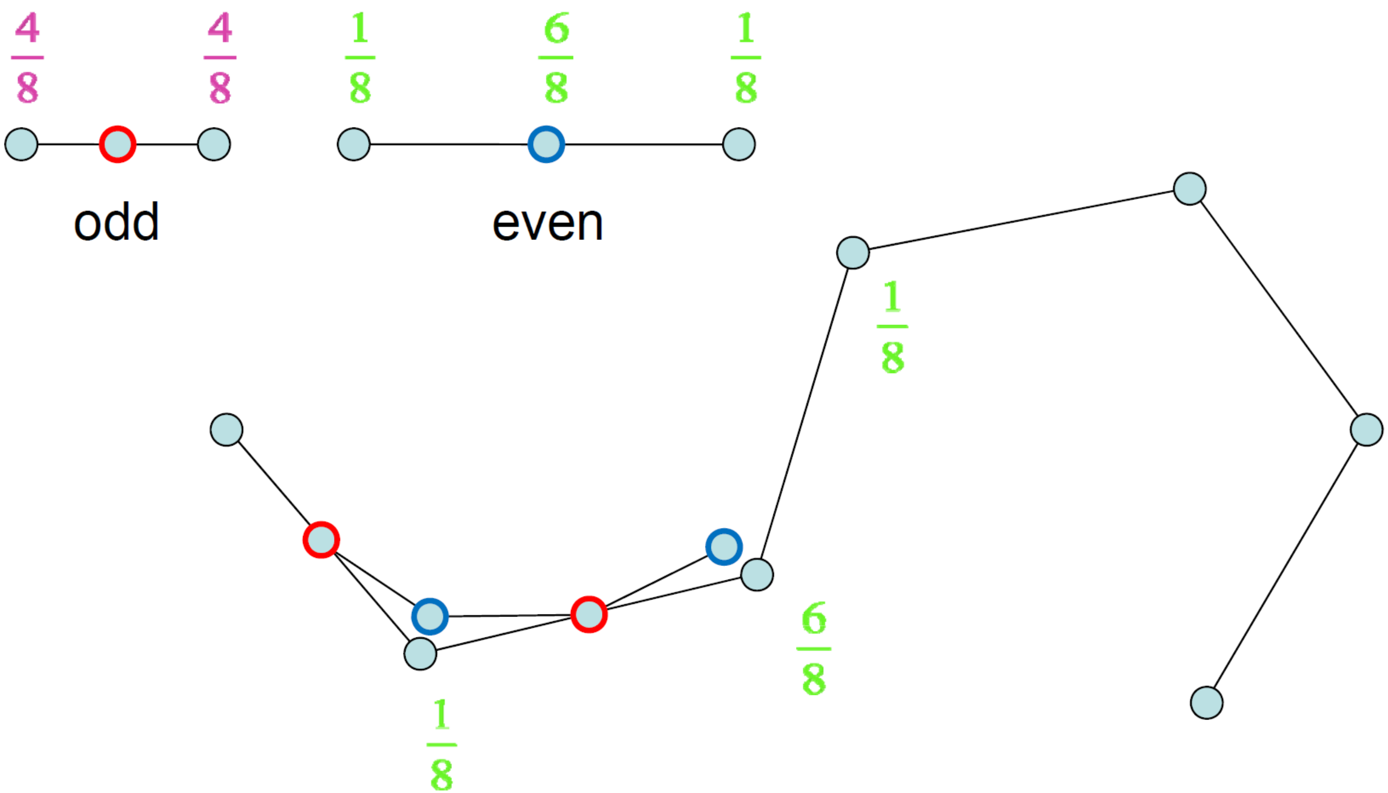

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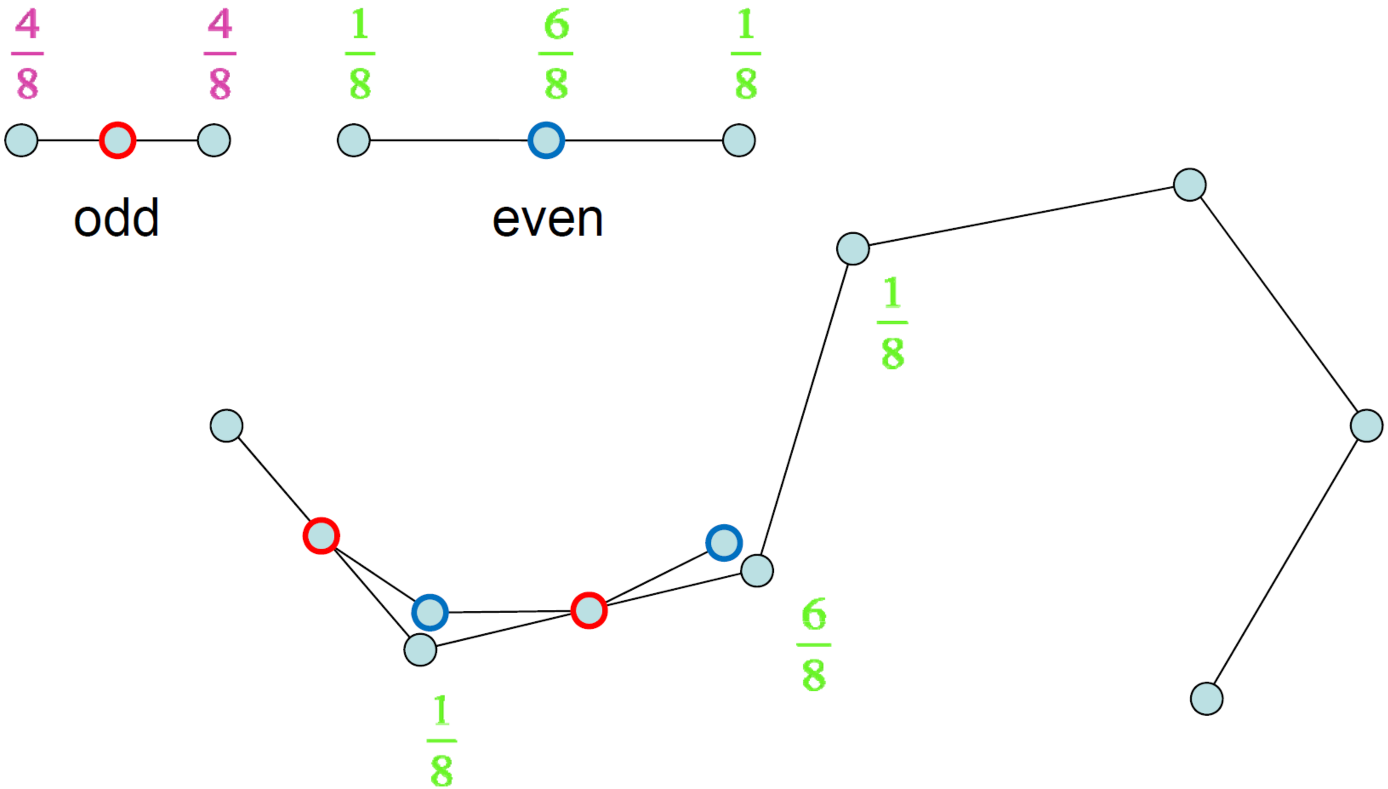
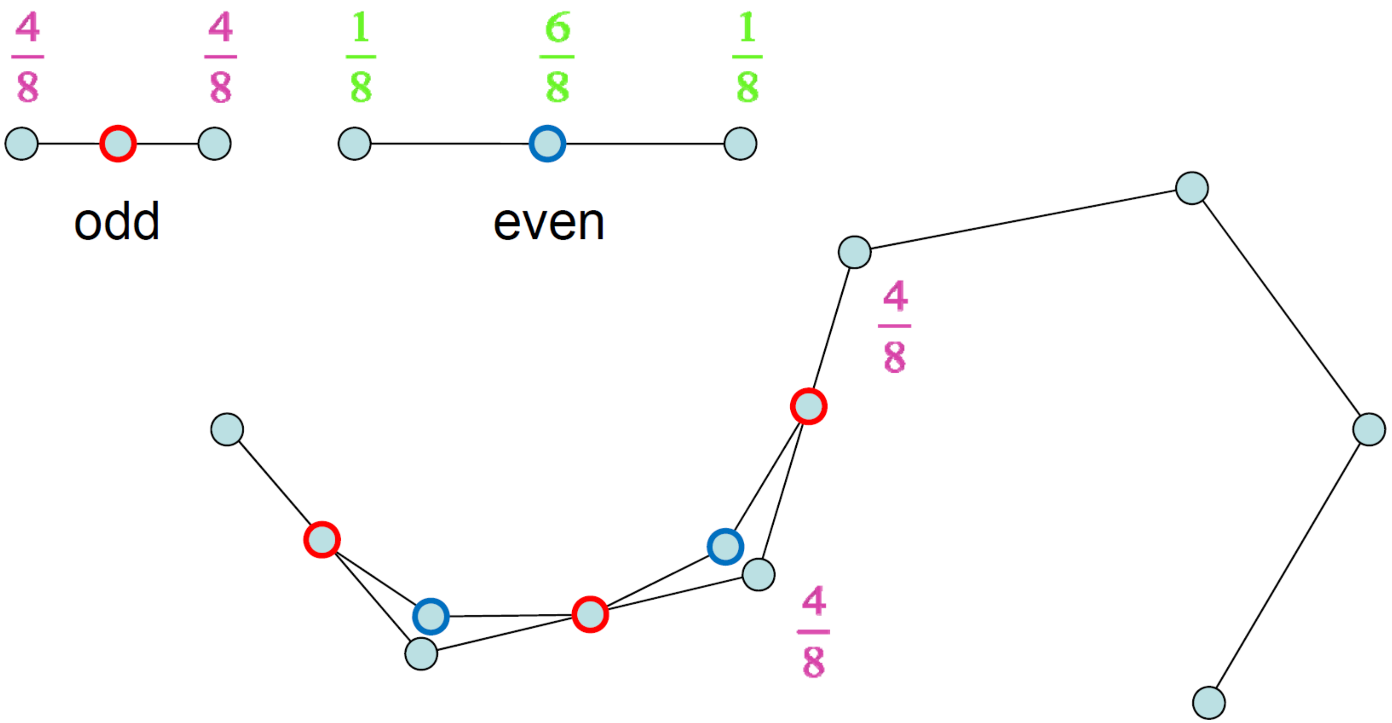
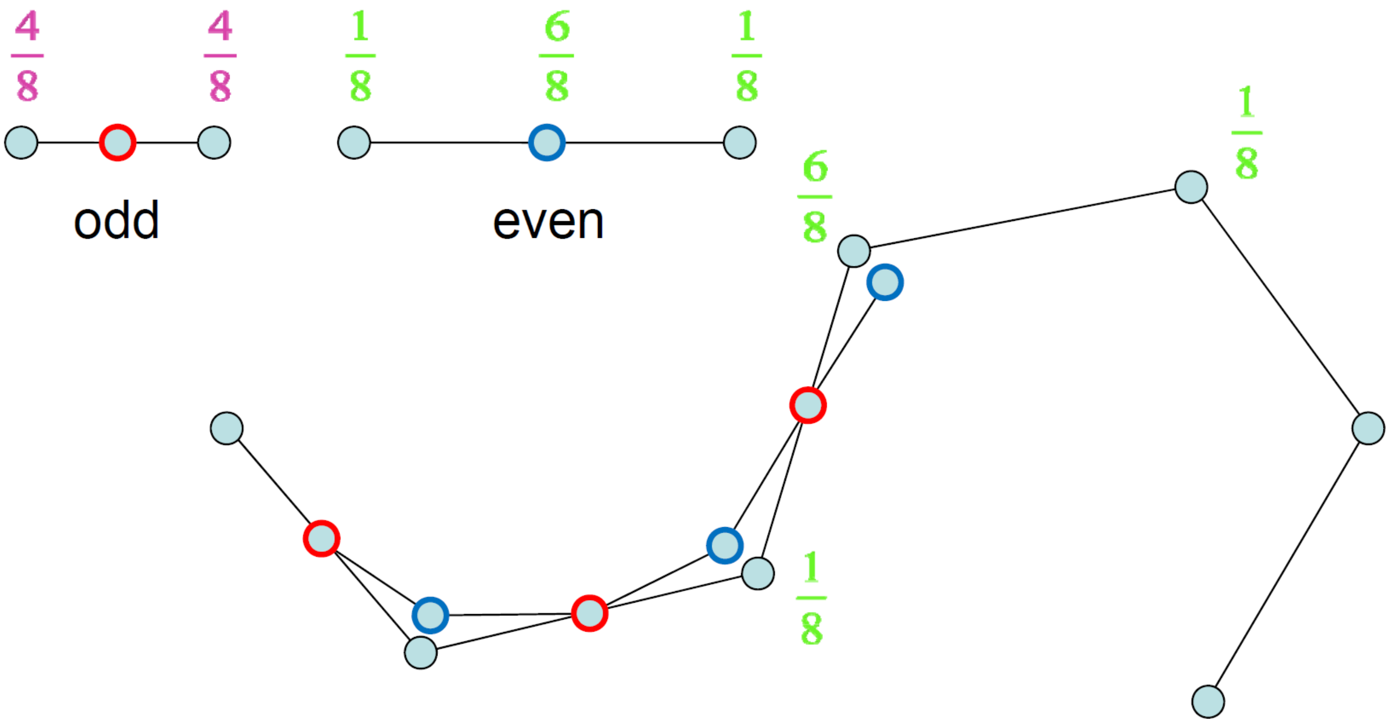


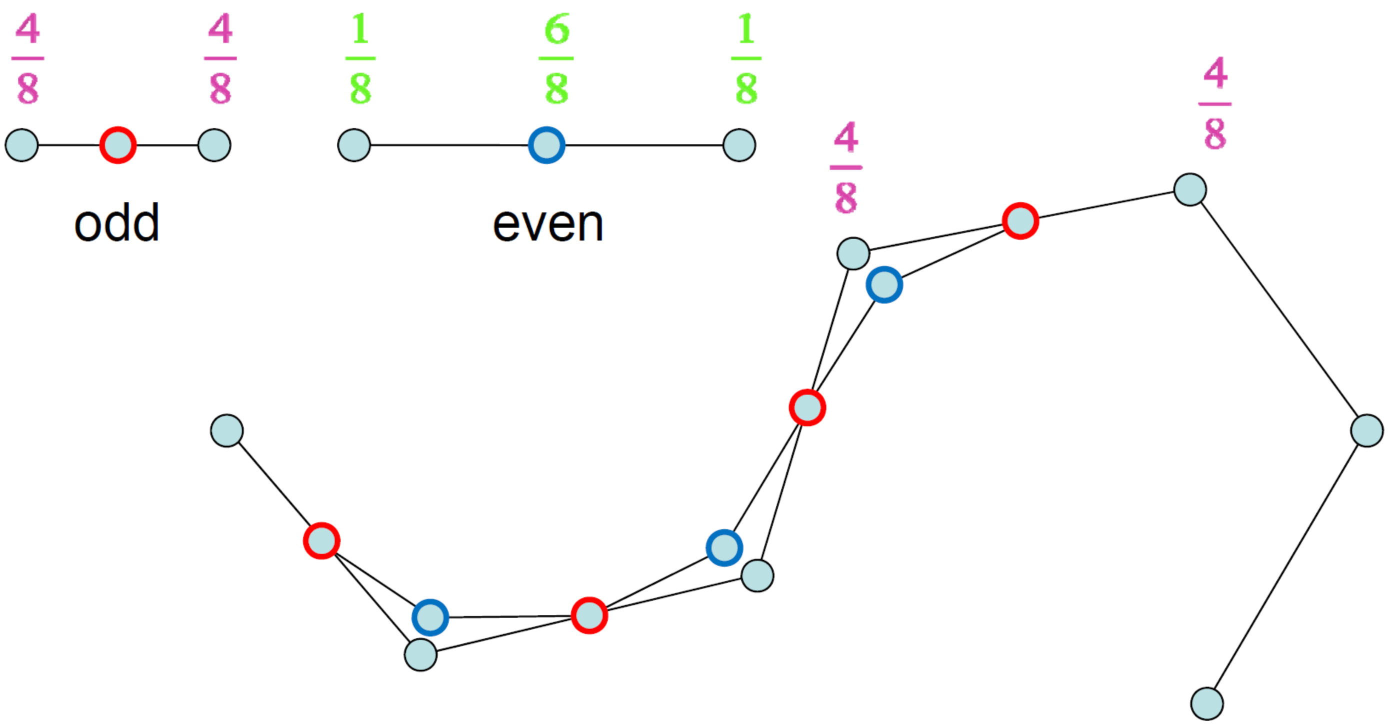
Cubic B-Spline



Cubic B-Spline



Cubic B-Spline



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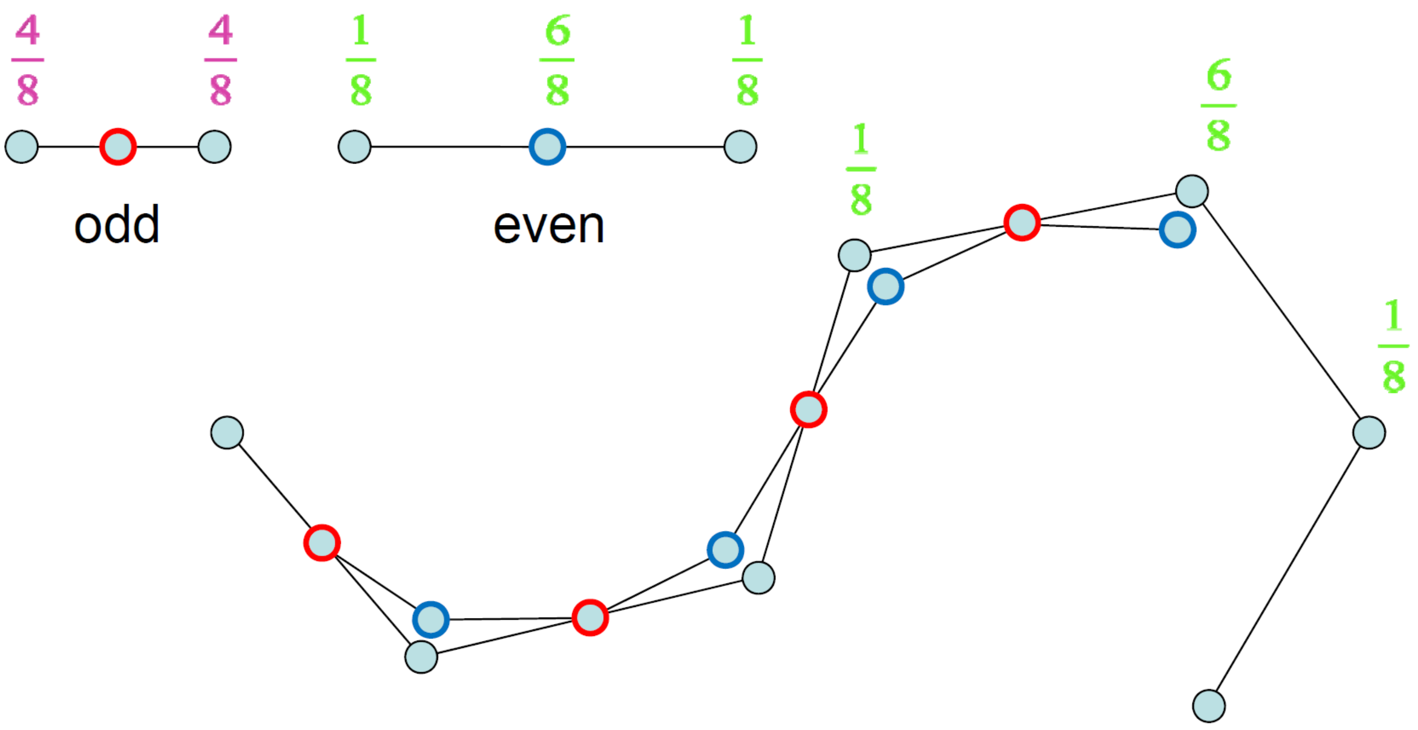

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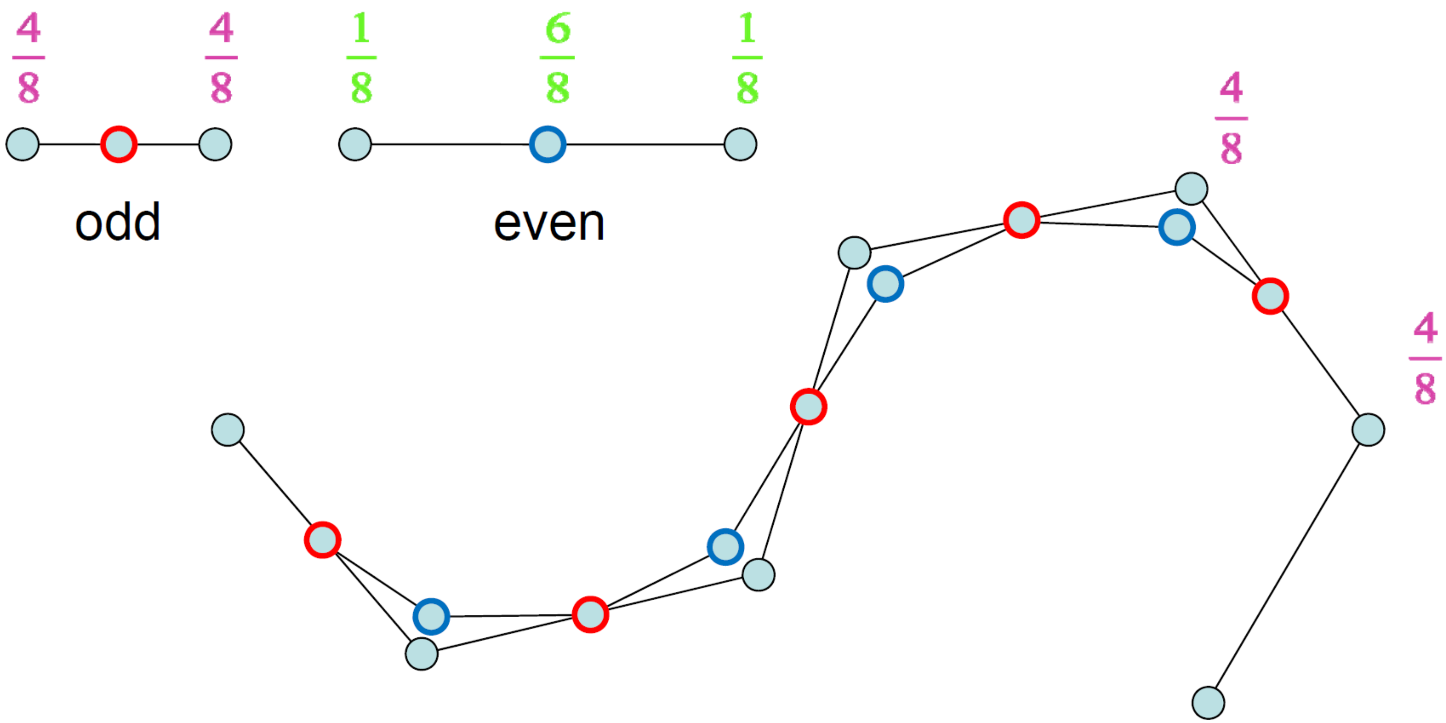

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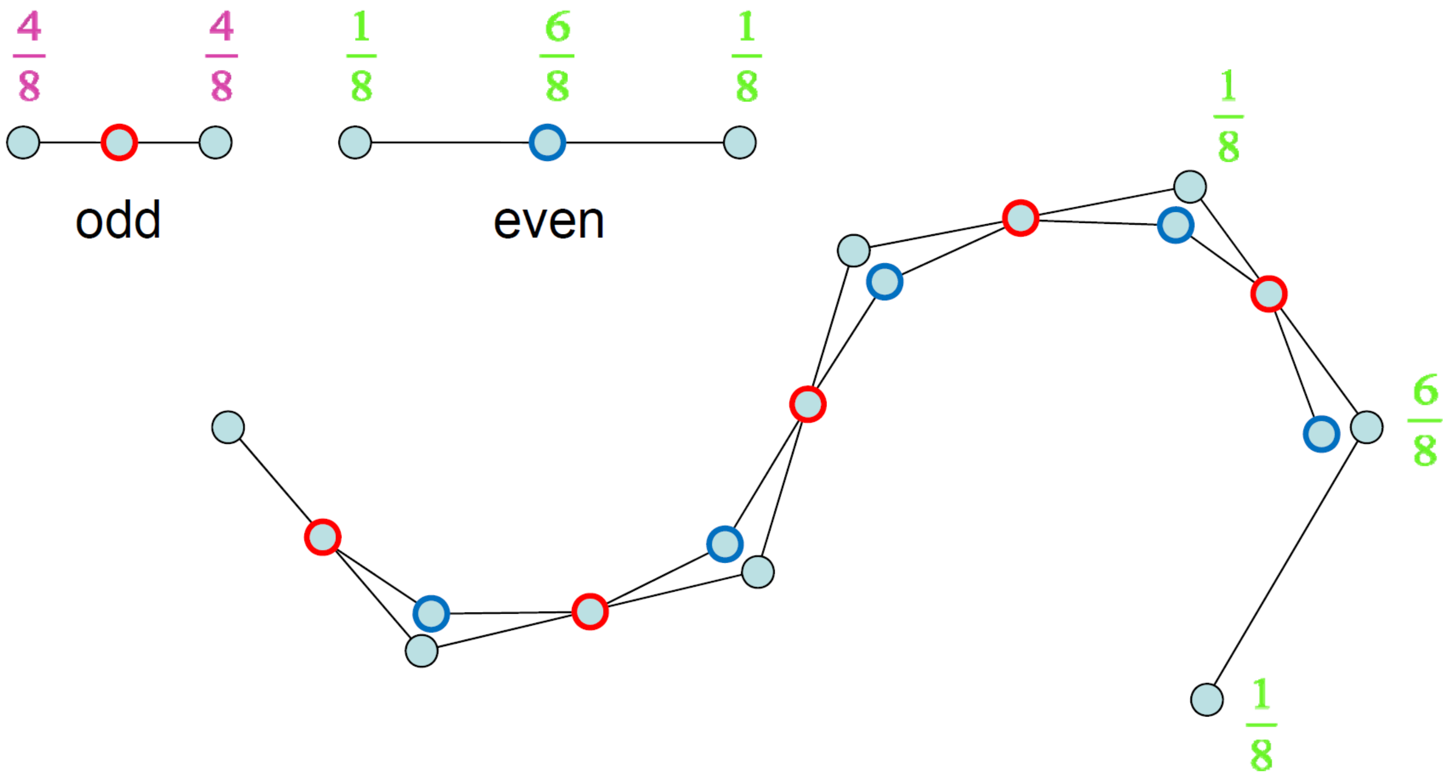

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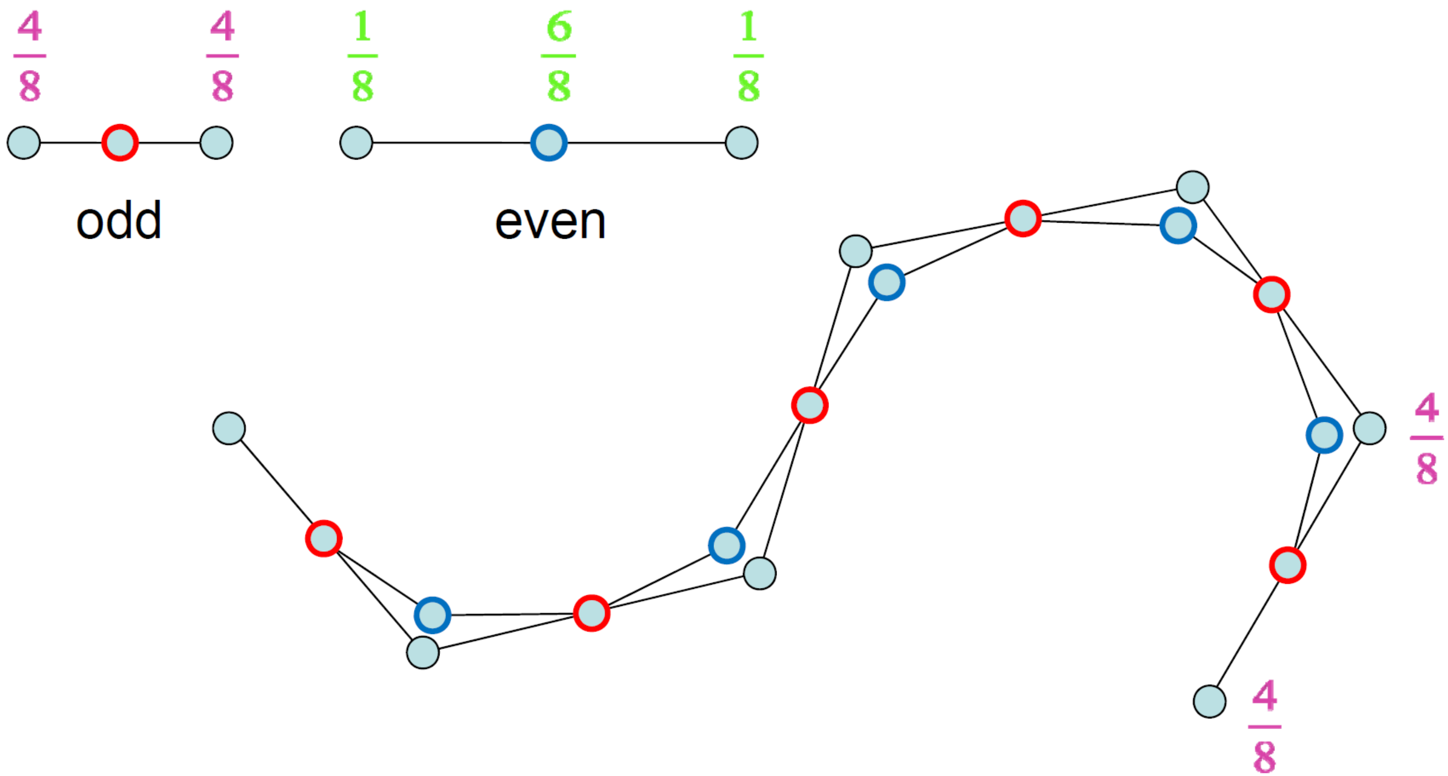
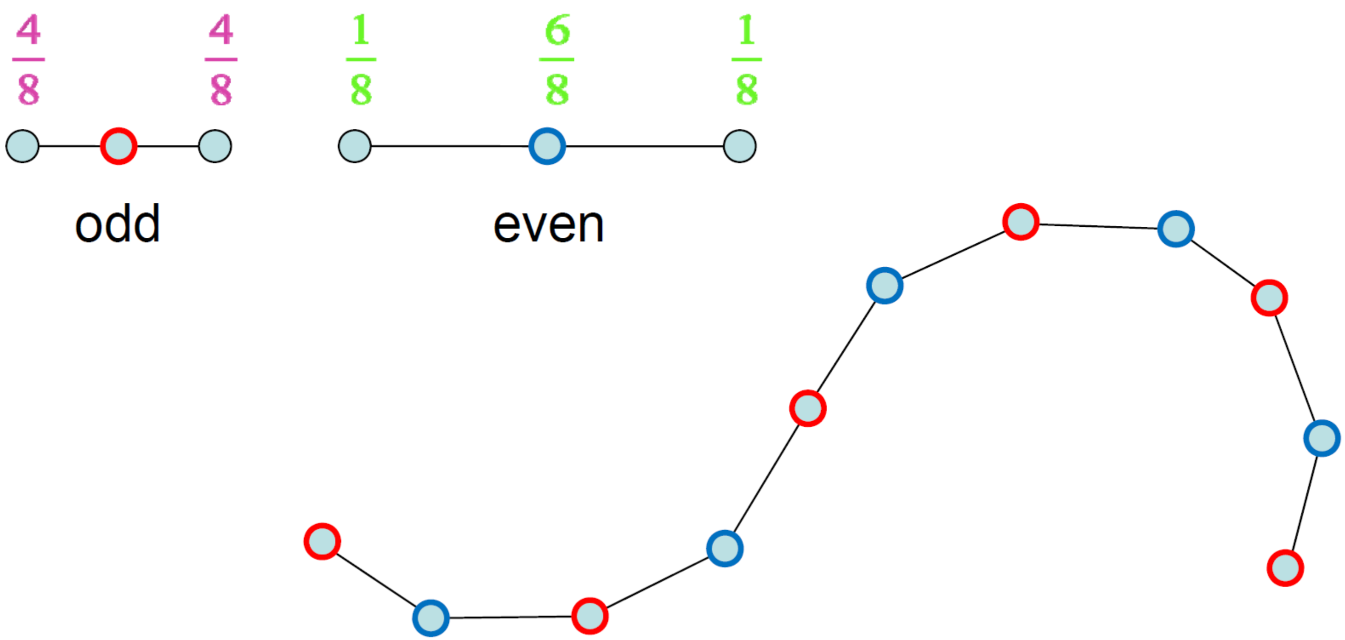


Cubic B-Spline



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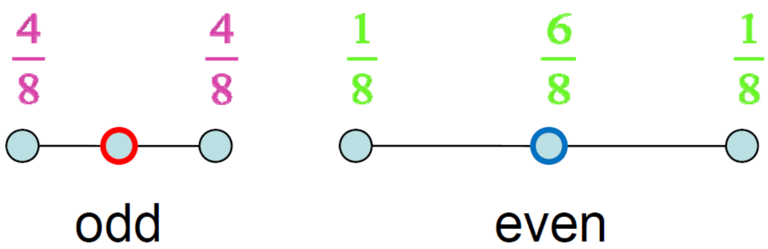
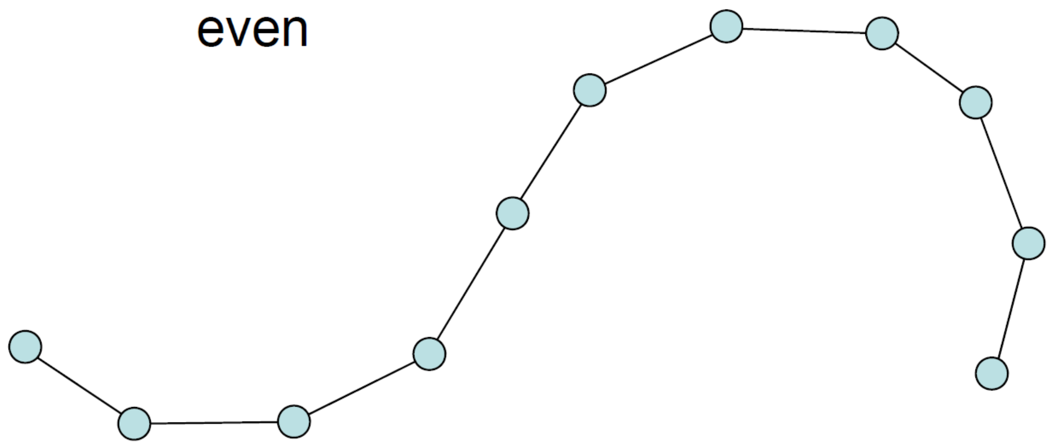




B-Spline Curves

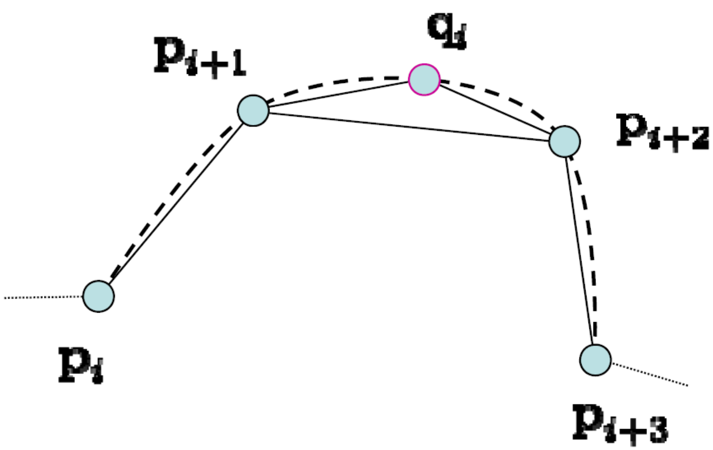
Subdivision rules for control polygon

$$\mathbf{d}^0 \rightarrow \mathbf{d}^1 = S\mathbf{d}^0 \rightarrow \ldots \rightarrow \mathbf{d}^j = S\mathbf{d}^{j-1} = S^j\mathbf{d}^0$$

Mask of size *n* yields $C^{n\text{-}1}$ curve

Interpolating (4-point scheme)

- Keep old vertices
- Generate new vertices by fitting cubic curve to old vertices
- C^1 continuous limit curve

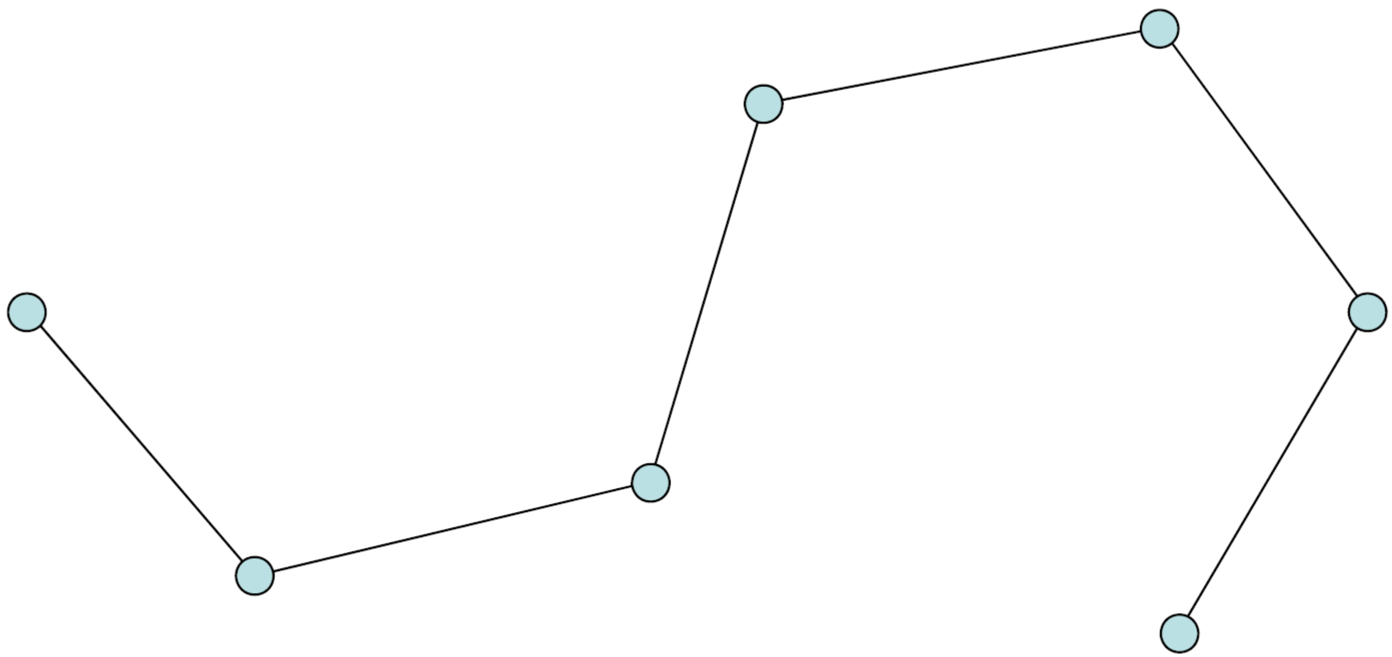


$$f(x) = ax^3 + bx^2 + cx + d$$

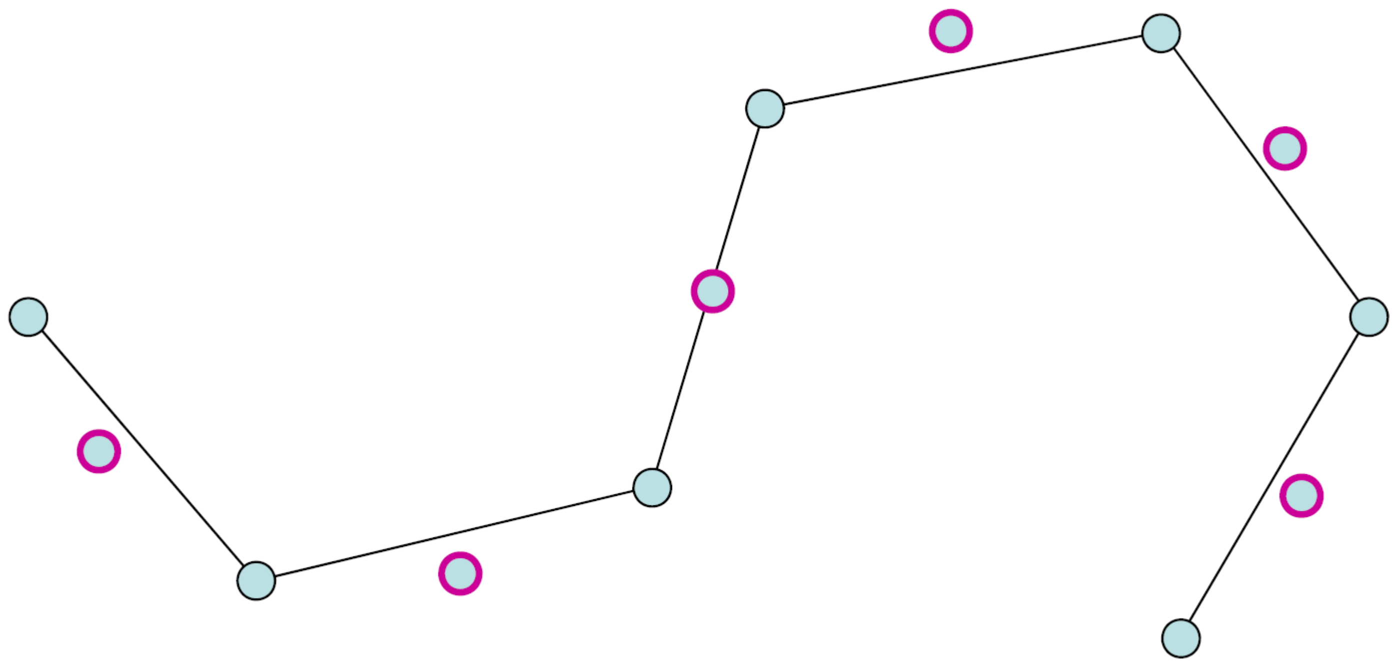
$$f(j) = \mathbf{p}_{i+j}\,, \quad j = 0, \ldots, 3$$

$$\begin{aligned}\mathbf{q}_i &= f(3/2) \\ &= \frac{1}{16}\left(-\mathbf{p}_i + 9\mathbf{p}_{i+1} + 9\mathbf{p}_{i+2} - \mathbf{p}_{i+3}\right)\end{aligned}$$

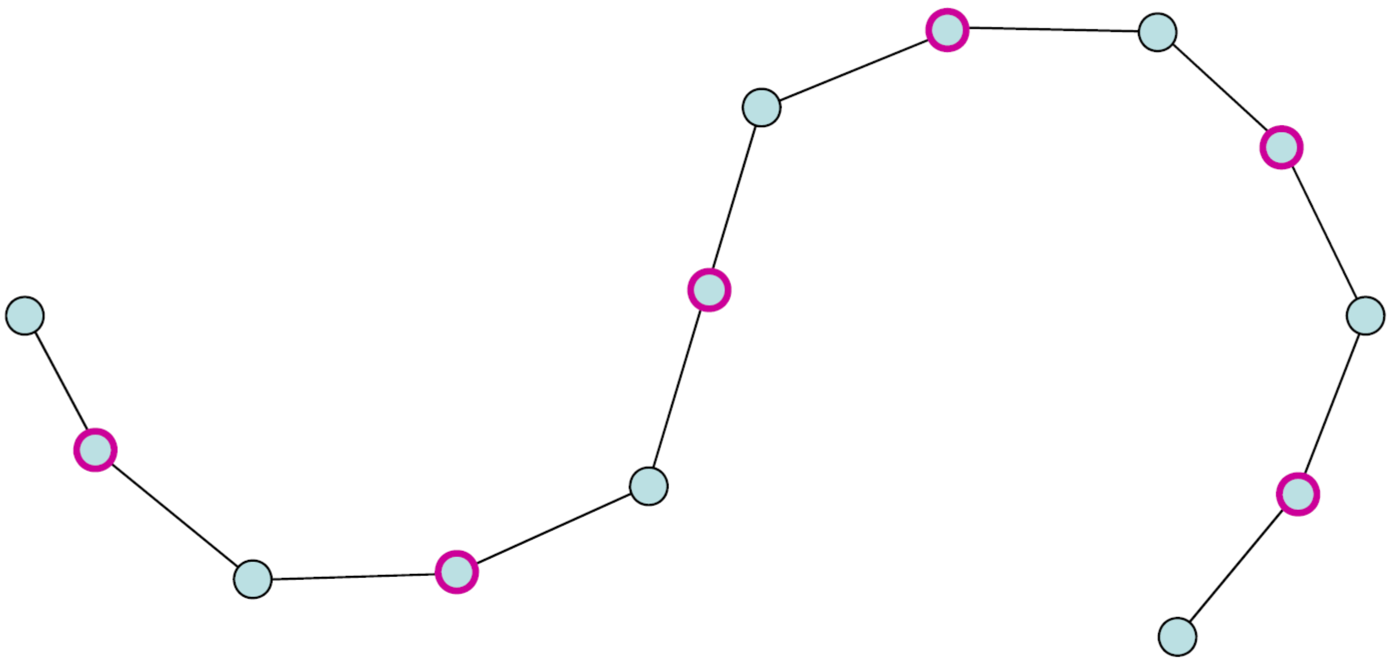
Interpolating



Interpolating



Interpolating



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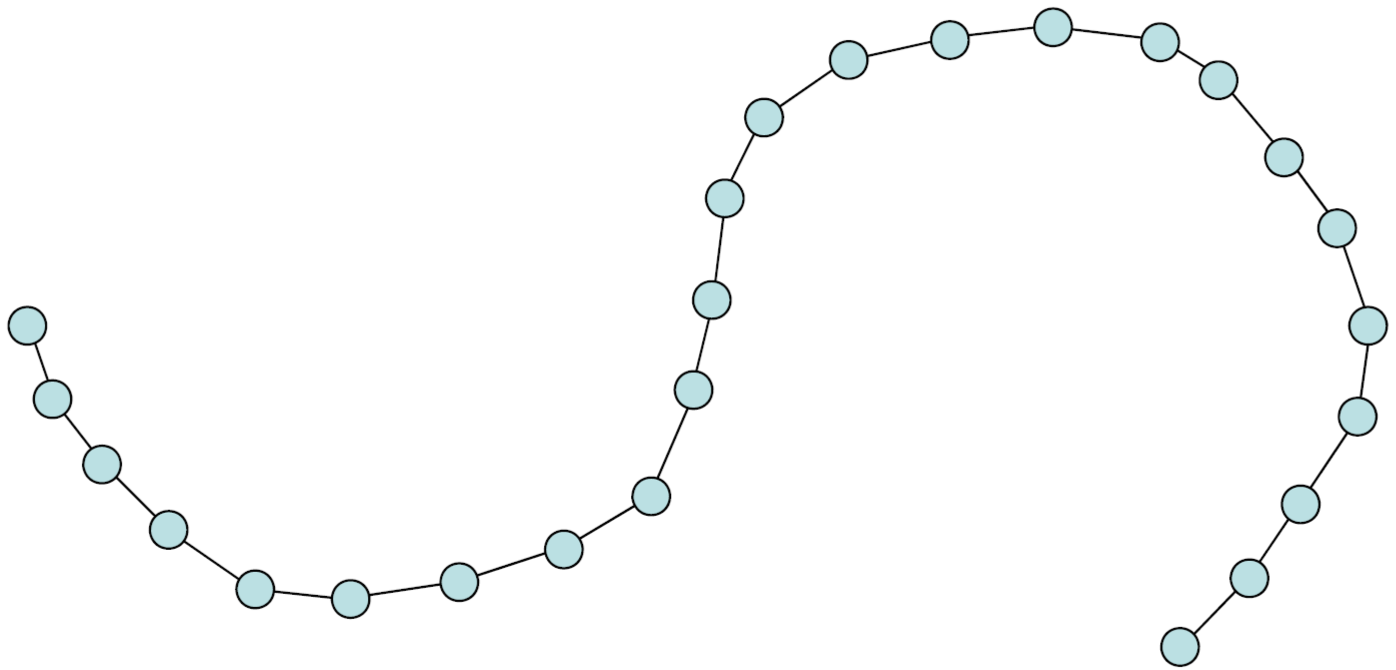
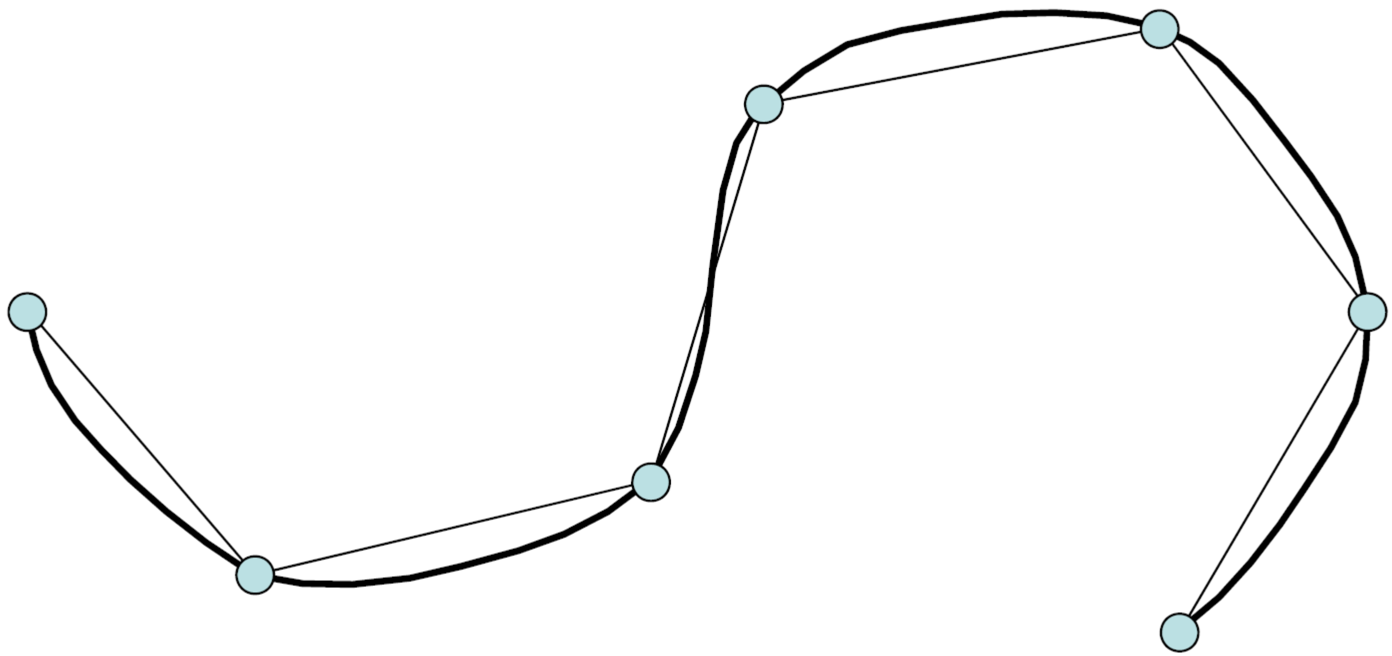


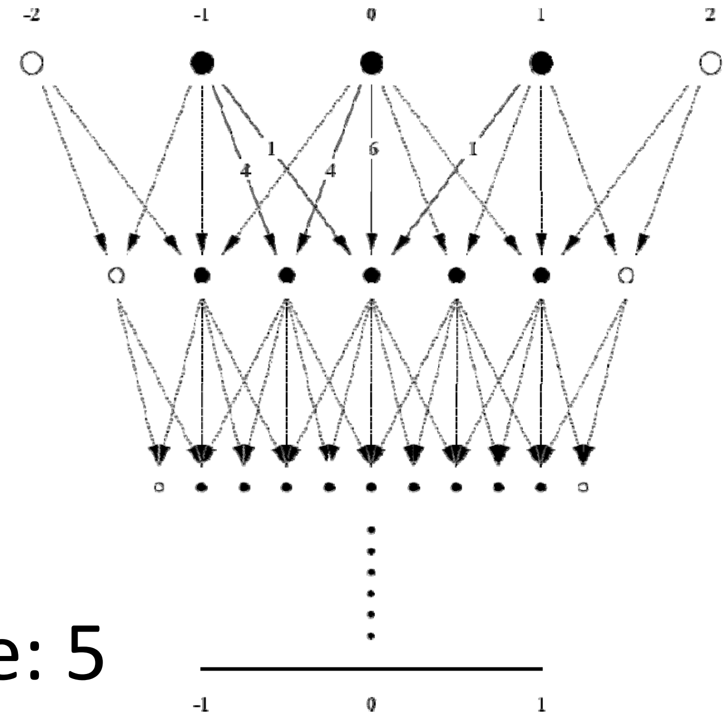
Interpolating



Local Subdivision Matrix

- Example: Cubic B-Splines

$$\begin{pmatrix} \mathbf{p}_{-2}^{j+1} \\ \mathbf{p}_{-1}^{j+1} \\ \mathbf{p}_{0}^{j+1} \\ \mathbf{p}_{1}^{j+1} \\ \mathbf{p}_{2}^{j+1} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_{-2}^{j} \\ \mathbf{p}_{-1}^{j} \\ \mathbf{p}_{0}^{j} \\ \mathbf{p}_{1}^{j} \\ \mathbf{p}_{2}^{j} \end{pmatrix}$$



- Invariant neighborhood size: 5

Analysis of Subdivision

- Analysis via eigen-decomposition of matrix *S*
  - Compute the eigenvalues

$$\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$$

  - and eigenvectors

$$X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$$

- Let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be real and *X* a complete set of eigenvectors

Limit Behavior - Position

- Any vector is linear combination of eigenvectors:

$$\mathbf{p} = \sum_{i=0}^{n-1} a_i \mathbf{x}_i \qquad a_i = \tilde{\mathbf{x}}_i^T \mathbf{p}$$

rows of X^{-1}

- Apply subdivision matrix:

$$S\mathbf{p}^0 = S\sum_{i=0}^{n-1} a_i \mathbf{x}_i = \sum_{i=0}^{n-1} a_i S\mathbf{x}_i = \sum_{i=0}^{n-1} a_i \lambda_i \mathbf{x}_i$$

Limit Behavior - Position

For convergence we need $1 = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1}$

Limit vector:

$$\mathbf{p}^\infty = \lim_{j\to\infty} S^j \mathbf{p}^0 = \lim_{j\to\infty} \sum_{i=0}^{n-1} a_i \lambda_i^j \mathbf{x}_i = a_0 \cdot \mathbf{1}$$

$\mathbf{p}_i^\infty = a_0 = \tilde{\mathbf{x}}_0^T \mathbf{p}^j$ independent of *j* !

Limit Behavior - Tangent

- Set origin at a_0:

$$\mathbf{p}^j = \sum_{i=1}^{n-1} a_i \lambda_i^j \mathbf{x}_i$$

- Divide by λ_1^j

$$\frac{1}{\lambda_1^j}\mathbf{p}^j = a_1 \mathbf{x}_1 + \sum_{i=2}^{n-1} a_i \left(\frac{\lambda_i}{\lambda_1}\right)^j \mathbf{x}_i$$

- Limit tangent given by:

$$\mathbf{t}_i^{\infty} = a_1 = \tilde{\mathbf{x}}_1^T \mathbf{p}^j$$

Limit Behavior - Tangent

All eigenvalues of S except $\lambda_0=1$ should be less than λ_1 to ensure existence of a tangent, i.e.

$$1 = \lambda_0 > \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{n-1}$$

Example: Cubic Splines

- Subdivision matrix & rules

$$S = \frac{1}{8}\begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{p}_{2i}^{j+1} &= \frac{1}{8}\mathbf{p}_{i-1}^{j} + \frac{6}{8}\mathbf{p}_{i}^{j} + \frac{1}{8}\mathbf{p}_{i+1}^{j} \\ \mathbf{p}_{2i+1}^{j+1} &= \frac{1}{2}\mathbf{p}_{i}^{j} + \frac{1}{2}\mathbf{p}_{i+1}^{j} \end{aligned}$$

- Eigenvalues

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4,) = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

Example: Cubic Splines

- Eigenvectors

$$X = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & -\frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 \\ 0 & -1 & 0 & 1 & 0 \\ & & \cdots & & \\ & & \cdots & & \\ & & \cdots & & \end{pmatrix}$$

- Limit position and tangent

$$\mathbf{p}_i^\infty = \tilde{\mathbf{x}}_0^T \mathbf{p}^j = \frac{1}{6}\left(\mathbf{p}_{i-1}^j + 4\mathbf{p}_i^j + \mathbf{p}_{i+1}^j\right)$$

$$\mathbf{t}_i^\infty = \tilde{\mathbf{x}}_1^T \mathbf{p}^j = \mathbf{p}_{i+1}^j - \mathbf{p}_i^j$$

Questions?