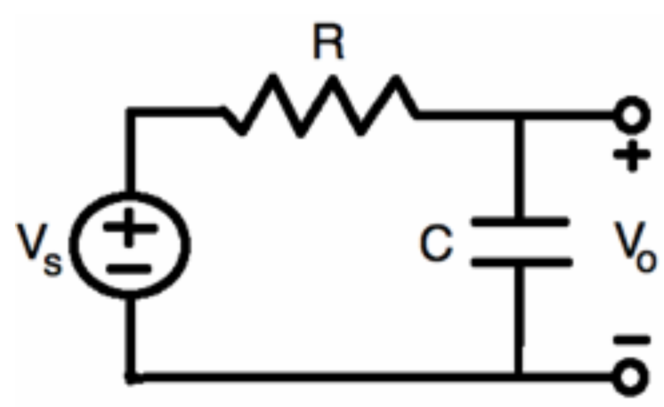
ACL2(r) Formalization of Fourier Series' Properties

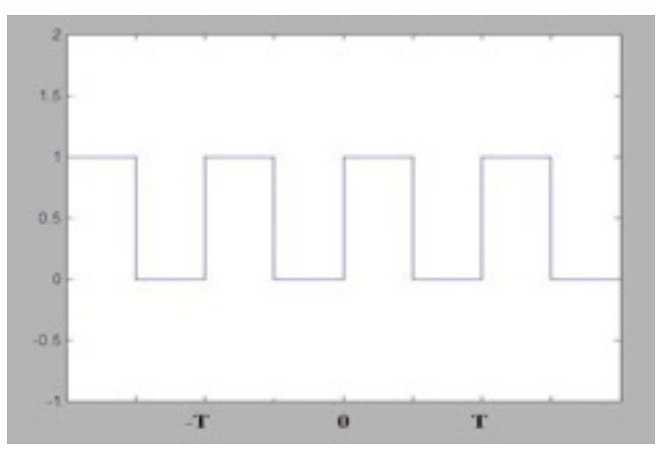
Cuong Chau

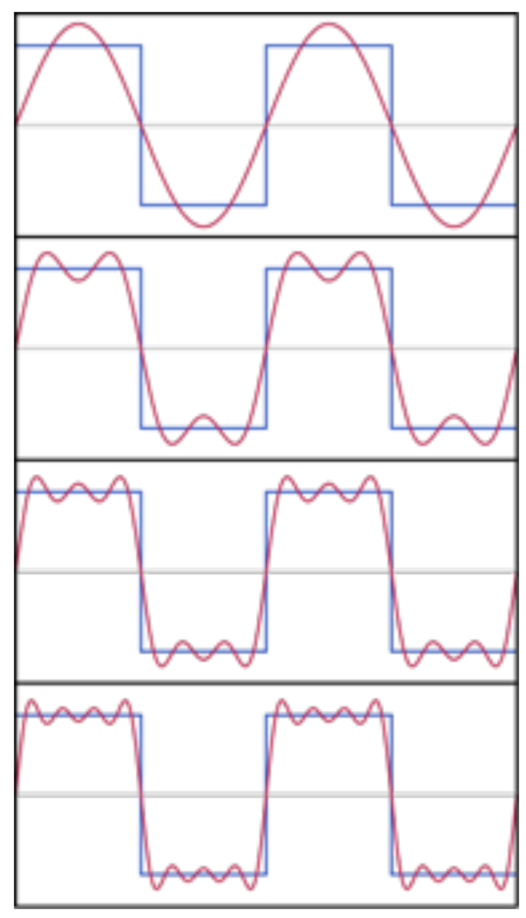
Department of Computer Science

The University of Texas at Austin

ckcuong@cs.utexas.edu







If the source voltage V_S is sinusoidal, the impedance Z_C of the capacitor is constant.

Then, the output voltage V_O can be computed as follows:

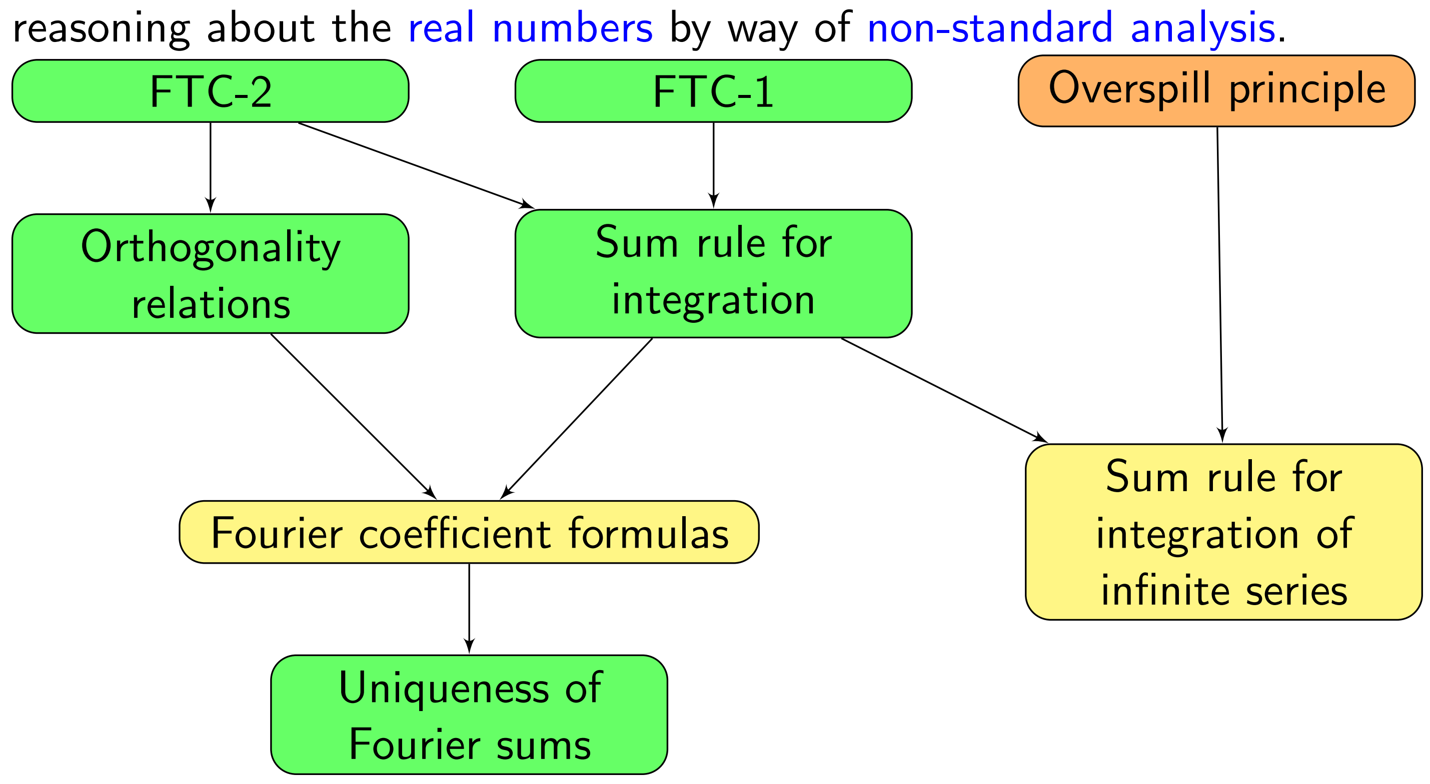
$$V_o = \frac{V_s Z_c}{R + Z_c}$$

Motivation

- Fourier series have many applications to a wide variety of mathematical and physical problems, electrical engineering, signal processing, etc.
- Solving differential equations is also a powerful application of Fourier analysis.
- We are interested in formalizing Fourier series (and possibly, Fourier transform) in ACL2 as a useful tool for formally analyzing analog circuits, mixed-signal integrated circuits, hybrid systems, etc.

Overview

We present our efforts in formalizing some basic properties of Fourier series in the logic of ACL2(r), which is a variant of ACL2 that supports reasoning about the real numbers by way of non-standard analysis.



Non-Standard Analysis

Formulate the operations of calculus using a logically rigorous notion of infinitesimal numbers, instead of epsilon-delta definition of limit.

Two basic approaches to the foundations:

1. Extend the reals to a bigger set of hyperreals, which includes infinitesimals [A. Robinson, 1996].
2. Nelson's Internal Set Theory views the "reals" as "all the reals", including infinitesimals, and considers a subset of standard reals [E. Nelson, 1977].

ACL2(r) follows (2).

Theorem 1 (Fourier coefficient formulas)

Consider the following Fourier sum for a periodic function with period 2L:

$$f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\tfrac{\pi}{L}x) + b_n \sin(n\tfrac{\pi}{L}x)\right)$$

Then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\frac{\pi}{L}x)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\frac{\pi}{L}x)dx.$$

Sum Rule for Definite Integrals of Infinite Series

Formalizing the sum rule for definite integrals of infinite series under certain conditions.

$$\int_a^b \lim_{N\to\infty}\left(\sum_{n=0}^{N} f_n(x)\right)dx \overset{?}{=} \lim_{N\to\infty}\left(\sum_{n=0}^{N}\int_a^b f_n(x)dx\right)$$

In non-standard analysis,

$$\int_a^b \text{st}\left(\sum_{n=0}^{H_0} f_n(x)\right)dx \overset{?}{=} \text{st}\left(\sum_{n=0}^{H_1}\int_a^b f_n(x)dx\right)$$

for all infinitely large natural numbers H_0 and H_1, where st is the standard-part function in non-standard analysis.

Sum Rule for Definite Integrals of Infinite Series

Requirement: A sequence of partial sums of real-valued continuous functions ***converges uniformly*** to a ***continuous limit function*** on the interval of interest. We come up with this requirement in two ways corresponding to two different conditions:

- Condition 1: A monotone sequence of partial sums of real-valued continuous functions ***converges pointwise*** to a ***continuous limit function*** on the closed and bounded interval of interest.
- Condition 2: A sequence of partial sums of real-valued continuous functions ***converges uniformly*** to a ***limit function*** on the interval of interest.

Conclusions

Extend a framework for formally evaluating definite integrals of real-valued continuous functions using FTC-2, even when functions contain free arguments.

Fourier coefficient formulas and the sum rule for definite integrals of infinite series have been formalized in ACL2(r).

We are confident that our frameworks can be applied to future work on Fourier series and, more generally, continuous mathematics, to be carried out in ACL2(r).