Integrating Proxy Theories and Numeric Model Lifting for Floating-Point Arithmetic

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Why Floating-Point Arithmetic?

Floating-point (FP) = practical approximation of real numbers

- Finite representation on computers
- Dynamic range
- Speed, implementation in hardware

FP arithmetic different from Real arithmetic

IEEE 754 (2008) Standard says:

$$x \; op_{\mathbb{F}} \; y = round(x \; op_{\mathbb{R}} \; y)$$

Standard describes 5 rounding modes

FP arithmetic different from Real arithmetic

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$$x \; op_{\mathbb{F}} \; y = round(x \; op_{\mathbb{R}} \; y)$$

Standard describes 5 rounding modes

Examples of formulas satisfiable in FP:

- $x \oplus y = x \wedge y > 0$
- $x \oplus (y \oplus z) > (x \oplus y) \oplus z$
- $x \otimes (y \oplus z) > (x \otimes y \oplus x \otimes z)$

Floating-point reasoning: approaches

- Traditionally: theorem proving, abstract domains
- More recently: decision procedures
  - Examples: Mathsat, z3
  - Big win: witness generation
  - Technique: bit-blasting, bit-vectors
  - Limitation: leads to huge boolean encodings

Using Real Arithmetic Solver [POPL13]

Automatic Detection of Floating-Point Exceptions

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Using Reduced Precision FP [IJCAR14]

Solving FP formula f

- $f' = \texttt{reduce\_precision}(f)$
- while($f' \neq f$)
- if $\exists\sigma : \sigma \models f'$
- if $\sigma \models f$
- return σ
- else
- increase precision of f'

Example

Consider f: $(x \oplus y) \oplus z > x \oplus (y \oplus z)$

Solve instead: $(x \oplus_9 y) \oplus_9 z > x \oplus_9 (y \oplus_9 z)$

Satisfiable in FP_9 (as any bit-blaster will tell you):

$$x_0 = y_0 = 1.18, \quad z_0 = 1.97 * 10^{-3}$$

Problem: $f(x_0, y_0, z_0) \rightarrow$ false ! What now?

Proxy solution

- Proxy solution gets discarded[IJCAR14] if it does not work as is:
  - effort wasted

Can we use the proxy solution in some way?

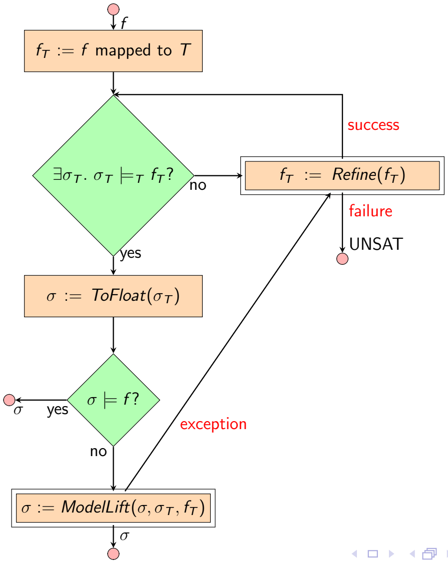
Can the proxy solution be lifted to an actual satisfying solution?

Lifting a proxy solution

Solving FP formula f

- $f' = \texttt{reduce\_precision}(f)$
- while($f' \neq f$)
-   if $\exists\sigma : \sigma \models f'$
-     if $\sigma \models f$
-       return σ
-     else
-       $\texttt{do\_something}(\sigma)$
-   else
-     increase precision of f'

Framework: Overview



Proxy theories for floating-point: Conditions

- offer a mapping *from* FP *formulas*
- easier to reason about than FP
- offer a mapping *to* FP *models*
- gradually refinable back to FP

Proxy theories for floating-point: Candidates

Reduced precision (reduced exponent + mantissa) FP

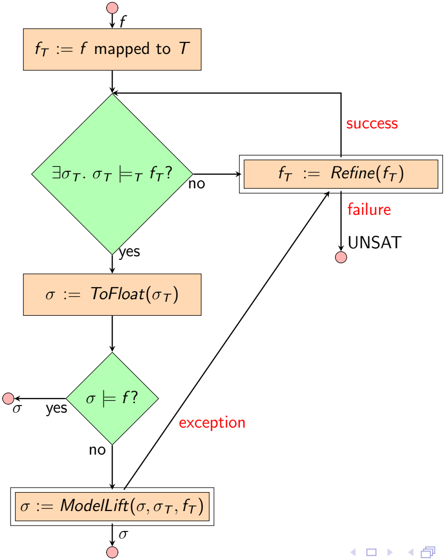
- "easier"
- map solutions to original precision FP by padding bits
- refine by gradually increasing exponent, mantissa

Real arithmetic

- sometimes easier
- map solutions to FP by rounding
- refine by interpreting some real operators as FP [DATE14]

Numeric Model Lifting

Framework: overview



Numeric Model Lifting

Assumption: proxy theory T delivers satisfying T assignment such that an FP solution is nearby

Idea for lifting proxy soln to FP soln:

- T assgn. gives satisfying Boolean skeleton
- fix constraints where T and FP disagree
- pick small subset $\texttt{Vars}(f)$ to do so, keep others constant

Numeric Model Lifting: Example

$$f(x,y) :: x \otimes y \otimes y \owedge 2 \vee x \oplus y \ocircle 0$$

- $\mathbb{R} : x * y^2 > 2 \vee x + y < 0 : \bar{x} = 1, \bar{y} = 1.42$
- In $\mathbb{R}$: $T \vee F = T$ In FP : $F \vee F = F$
- Goal: fix FP assgn. so that $x \otimes y \otimes y \owedge 2 = T$

$$f'(x) :: x \otimes \bar{\bar{y}} \otimes \bar{\bar{y}} \owedge 2 \wedge \neg(x \oplus \bar{\bar{y}} \ocircle 0)$$

f' is: (i) univariate, (ii) linear, (iii) conjunctive

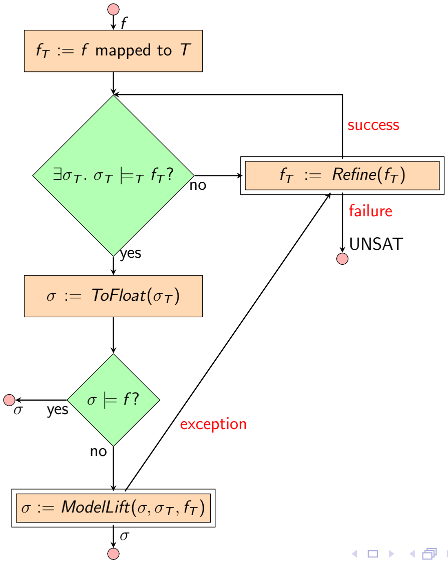
Numeric Model Lifting: Summary

1. Reduces decision problem (f) to simpler one (f')
2. Uses off-the-shelf floating-point SMT solver for f'

Benefits:

✓ propositional structure of f reduced to conjunction
✓ typically, $\mathrm{Vars}(f') \subsetneq \mathrm{Vars}(f)$
✓ often, degree $\deg(f') < \deg(f)$
✓ Independent of where proxy solution came from

Framework: On Soundness, Termination, Completeness



Experimental Evaluation

Experimental Setup

Set I:

- Non-linear benchmarks [FMSD14]
- Ignored casts (single precision), ignored special values
- Benchmarks are satisfiable or status is unknown

Set II:

- False identity non-linear benchmarks, $E - \widehat{E} > \epsilon$
  e.g., $(a^2 \ominus b^2) - (a \ominus b)(a \oplus b) > \epsilon$
- is of interest in compiler optimization
- single precision

Timeout : 20 min

Experimental Evaluation (Set II)

	MOLLY(RPFPA)			APPROX [IJCAR14]		MATHSAT
Problem	*It*	*Lifted?*	*Time (s)*	*It*	*Time (s)*	*Time (s)*
False Identity benchmarks						
23	3	✓	148.6	8	163.7	60.5
24	2	✓	64.6	8	137.9	108.4
25	8	×	162.7	8	137.2	108.4
26	1	✓	0.9	8	137.2	108.2
27	8	×	278.2	8	162.8	47.7
28	1	✓	12.4	8	123.1	51.8
29	4	×	70.2	4	9.8	112.4
30	2	✓	62.6	8	108.5	108.7
31	3	✓	144.5	8	172.4	122.5
32	3	✓	157.2	8	**TO**	133.6
33	1	✓	1.1	4	0.6	133.6
34	4	✓	181.4	8	**TO**	605.4
35	1	✓	2.1	8	7.7	596.5
36	1	×	0.1	1	0.1	0.3
37	3	×	0.5	3	0.5	0.3

Experimental Evaluation (Results)

Set I: total 22, Set II: total 15

		Molly	Approx	Mathsat
I	# Solved	14(9)	13	15
	Total Time(s)	3067	1650	6656
	Avg. Time(s)	219	127	443
	# TO	8	9	7
II	# Solved	15(10)	13	15
	Total Time(s)	1287	1161	2237
	Avg. Time(s)	86	89	149
	# TO	0	2	0

Future Directions

- Non-symbolic model lifting
- Numeric solvers for approximate solutions
- Handling other combinations of proxy $\leftrightarrow$ actual solutions
  - UNSAT $\leftrightarrow$ UNSAT
  - UNSAT $\leftrightarrow$ SAT
  - SAT $\leftrightarrow$ UNSAT

Thank You!

Backup Slides

Experimental Evaluation (Set I)

	MOLLY(RPFPA)			APPROX [IJCAR14]		MATHSAT
Problem	*It*	*Lifted?*	*Time (s)*	*It*	*Time (s)*	*Time (s)*
I. Non-linear benchmarks from [FMSD13]						
1	1	✓	7.8	2	5.0	344.0
2	1	✓	15.8	2	12.3	986.5
3	2	×	60.1	2	45.6	995.9
4	-	-	**TO**	-	**TO**	977.6
5	-	-	**TO**	-	**TO**	983.6
6	-	-	**TO**	-	**TO**	977.1
7	-	-	**TO**	-	**TO**	983.5
8	-	-	**TO**	-	**TO**	**TO**
9	8	×	337.1	8	330.8	**TO**
10	-	-	**TO**	-	**TO**	**TO**
11	1	✓	3.2	2	0.3	61.8
12	-	×	680.5	2	0.3	**TO**
13	7	✓	863.3	-	**TO**	**TO**
14	-	-	**TO**	-	**TO**	**TO**
15	-	-	**TO**	-	**TO**	**TO**
16	8	×	484.7	8	116.6	46.7
17	8	×	350.3	8	322.2	47.0
18	2	✓	4.9	6	29.4	46.8
19	2	✓	22.1	3	32.5	47.2
20	1	✓	3.3	2	6.3	46.5
21	2	✓	263.4	3	599.9	46.8
22	3	✓	39.1	4	118.8	65.7

Experimental setup

Instantiation with Real Arithmetic Proxy Theory
Set III:

- $E > \widehat{E}$

$$(((a_1 \oplus a_2) \oplus (a_3 \oplus a_4)) \oplus a_5) > ((((a_1 \oplus a_2) \oplus a_3) \oplus a_4) \oplus a_5)$$

- (0, 1024.0]
- single precision, RoundToNearestEven
- Offset O is singleton (gradient analysis)

Experimental Evaluation

Set III benchmarks

	Molly			Approx		Mathsat
#Vars	*It*	*Lifted?*	*Time (s)*	*It*	*Time (s)*	*Time (s)*
35	6	✓	30.5	15	153	81.6
40	3	✓	11.9	7	34	278.2
45	8	✓	448.6	33	**TO**	457.1
50	5	✓	25.1	20	344	164.5
55	5	✓	28.3	16	210	754.8
60	3	✓	17.2	34	**TO**	**TO**
65	7	✓	42.0	11	88	**TO**