Theorems from CDS4LTL (Expanded)

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Abstract

The first section of this document is a collection of the axioms and theorems of the propositional calculus in Gries and Schneider's book *A Logical Approach to Discrete Math*, Springer-Verlag, 1993 (LADM). The numbering is consistent with that text with the chapter number followed by the equation number separated by a period. Additional theorems, either not included in LADM or included but not numbered, are indicated by a three-part number with two period separators. The second section is a collection of the axioms and theorems of linear temporal logic in Warford, Vega, and Staley's paper *A Calculational Deductive System for Linear Temporal Logic* (CDS4LTL), Pepperdine University Research Report, Natural Science Division, 2019. And, the third section is a collection of the axioms and theorems of linear temporal logic in Staley's paper *A Calculational Deductive System for Linear Temporal Logic: Additional Theorems* (CDS4LTL), Pepperdine University Research Report, Natural Science Division, 2020.

Table of Precedences

$[x := e]$ (textual substitution)	Highest precedence
$\neg$ $\circ$ $\Diamond$ $\Box$	
$\mathcal{U}$ $\mathcal{W}$	
$=$ (conjunctional)	
$\vee$ $\wedge$	
$\Rightarrow$ $\Leftarrow$	
$\equiv$ (associative)	Lowest precedence

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Theorems of the Propositional Calculus

Equivalence and $true$

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \ \equiv \ (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and $false$

(3.8) **Definition of $false$:** $false \equiv \neg true$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \ \equiv \ \neg p \equiv q$

(3.10) **Definition of $\not\equiv$:** $(p \not\equiv q) \ \equiv \ \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of $false$:** $\neg false \equiv true$

(3.14) $(p \not\equiv q) \ \equiv \ \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv false$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \ \equiv \ (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \ \equiv \ (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \ \equiv \ (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \ \equiv \ p \equiv q \not\equiv r$

(3.19.1) $p \not\equiv p \not\equiv q \ \equiv \ q$

Disjunction

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$

(3.28) **Axiom, Excluded middle:** $p \vee \neg p$

(3.29) **Zero of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction

(3.35) **Axiom, Golden rule:** $p \wedge q \equiv p \equiv q \equiv p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge true \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge false \equiv false$

(3.41) **Distributivity of $\wedge$ over $\wedge$:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:**
(a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:**
(a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:**
(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$

(3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$

(3.50) $p \wedge (q \equiv p) \equiv p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

(3.52) **Equivalence:** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

Implication

(3.57) **Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$

(3.59) **Implication:** $p \Rightarrow q \equiv \neg p \vee q$

(3.60) **Implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:** $p \Rightarrow (q \equiv r) \equiv (p \Rightarrow q) \equiv (p \Rightarrow r)$

(3.63.1) **Distributivity of $\Rightarrow$ over $\wedge$:** $p \Rightarrow q \wedge r \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

(3.63.2) **Distributivity of $\Rightarrow$ over $\vee$:** $p \Rightarrow q \vee r \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$

(3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv true$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p$

(3.72) **Right zero of $\Rightarrow$:** $p \Rightarrow true \equiv true$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \equiv p$

(3.74) $p \Rightarrow false \equiv \neg p$

(3.74.1) $\neg p \Rightarrow false \equiv p$

(3.75) $false \Rightarrow p \equiv true$

(3.76) **Weakening/strengthening:**

(a) $p \Rightarrow p \vee q$ (Weakening the consequent)

(b) $p \wedge q \Rightarrow p$ (Strengthening the antecedent)

(c) $p \wedge q \Rightarrow p \vee q$ (Weakening/strengthening)

(d) $p \vee (q \wedge r) \Rightarrow p \vee q$

(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.76.1) $p \wedge q \Rightarrow p \vee r$ (Weakening/strengthening)

(3.76.2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

(3.76.3) $(p \vee q) \wedge (q \Rightarrow r) \Rightarrow p \vee r$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.77.1) **Modus tollens:** $(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p$

(3.77.2) $((p \Rightarrow q) \Rightarrow (r \Rightarrow s)) \wedge (s \Rightarrow t) \Rightarrow ((p \Rightarrow q) \Rightarrow (r \Rightarrow t))$

(3.77.3) $((p \Rightarrow (q \Rightarrow r)) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow (q \Rightarrow s))$

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv p \vee q \Rightarrow r$

(3.78.1) $(p \Rightarrow r) \vee (q \Rightarrow r) \equiv p \wedge q \Rightarrow r$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:**

(a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

(3.82.1) **Transitivity of $\equiv$:** $(p \equiv q) \wedge (q \equiv r) \Rightarrow (p \equiv r)$

(3.82.2) $(p \equiv q) \Rightarrow (p \Rightarrow q)$

Leibniz as an axiom

This section uses the following notation: E^z_X means $E[z := X]$.

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E^z_e = E^z_f$

(3.84) **Substitution:**

(a) $(e = f) \wedge E^z_e \;\equiv\; (e = f) \wedge E^z_f$

(b) $(e = f) \Rightarrow E^z_e \;\equiv\; (e = f) \Rightarrow E^z_f$

(c) $q \wedge (e = f) \Rightarrow E^z_e \;\equiv\; q \wedge (e = f) \Rightarrow E^z_f$

(3.85) **Replace by $true$:**

(a) $p \Rightarrow E^z_p \;\equiv\; p \Rightarrow E^z_{true}$

(b) $q \wedge p \Rightarrow E^z_p \;\equiv\; q \wedge p \Rightarrow E^z_{true}$

(3.86) **Replace by $false$:**

(a) $E^z_p \Rightarrow p \;\equiv\; E^z_{false} \Rightarrow p$

(b) $E^z_p \Rightarrow p \vee q \;\equiv\; E^z_{false} \Rightarrow p \vee q$

(3.87) **Replace by $true$:** $p \wedge E^z_p \;\equiv\; p \wedge E^z_{true}$

(3.88) **Replace by $false$:** $p \vee E^z_p \;\equiv\; p \vee E^z_{false}$

(3.89) **Shannon:** $E^z_p \;\equiv\; (p \wedge E^z_{true}) \vee (\neg p \wedge E^z_{false})$

(3.89.1) $E^z_{true} \wedge E^z_{false} \Rightarrow E^z_p$

Additional theorems concerning implication

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

Proof technique metatheorems.

(4.4) **Deduction (assume conjuncts of antecedent):**

To prove $P_1 \wedge P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.

You cannot use textual substitution in P_1 or P_2.

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.7.1) **Truth implication:** To prove P, prove $true \Rightarrow P$.

(4.9) **Proof by contradiction:** To prove P, prove $\neg P \Rightarrow false$.

(4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.

Theorems of Linear Temporal Logic

Next $\circ$

(1) **Axiom, Self-dual:** $\circ \neg p \equiv \neg \circ p$

(2) **Axiom, Distributivity of $\circ$ over $\Rightarrow$:** $\circ (p \Rightarrow q) \equiv \circ p \Rightarrow \circ q$

(3) **Linearity:** $\circ p \equiv \neg \circ \neg p$

(4) **Distributivity of $\circ$ over $\vee$:** $\circ (p \vee q) \equiv \circ p \vee \circ q$

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ (p \wedge q) \equiv \circ p \wedge \circ q$

(6) **Distributivity of $\circ$ over $\equiv$:** $\circ (p \equiv q) \equiv \circ p \equiv \circ q$

(7) **Truth of $\circ$:** $\circ true \equiv true$

(8) **Falsehood of $\circ$:** $\circ false \equiv false$

Until $\mathcal{U}$

(9) **Axiom, Distributivity of $\circ$ over $\mathcal{U}$:** $\circ (p \,\mathcal{U}\, q) \equiv \circ p \,\mathcal{U} \circ q$

(10) **Axiom, Expansion of $\mathcal{U}$:** $p \,\mathcal{U}\, q \equiv q \vee (p \wedge \circ (p \,\mathcal{U}\, q))$

(11) **Axiom, Right zero of $\mathcal{U}$:** $p \,\mathcal{U}\, false \equiv false$

(12) **Axiom, Left distributivity of $\mathcal{U}$ over $\vee$:** $p \,\mathcal{U}\, (q \vee r) \equiv p \,\mathcal{U}\, q \vee p \,\mathcal{U}\, r$

(13) **Axiom, Right distributivity of $\mathcal{U}$ over $\vee$:** $p \,\mathcal{U}\, r \vee q \,\mathcal{U}\, r \Rightarrow (p \vee q) \,\mathcal{U}\, r$

(14) **Axiom, Left distributivity of $\mathcal{U}$ over $\wedge$:** $p \,\mathcal{U}\, (q \wedge r) \Rightarrow p \,\mathcal{U}\, q \wedge p \,\mathcal{U}\, r$

(15) **Axiom, Right distributivity of $\mathcal{U}$ over $\wedge$:** $(p \wedge q)\,\mathcal{U}\,r \equiv p\,\mathcal{U}\,r \wedge q\,\mathcal{U}\,r$

(16) **Axiom, $\mathcal{U}$ implication ordering:** $p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,r$

(17) **Axiom, Right $\mathcal{U} \vee$ ordering:** $p\,\mathcal{U}\,(q\,\mathcal{U}\,r) \Rightarrow (p \vee q)\,\mathcal{U}\,r$

(18) **Axiom, Right $\wedge\,\mathcal{U}$ ordering:** $p\,\mathcal{U}\,(q \wedge r) \Rightarrow (p\,\mathcal{U}\,q)\,\mathcal{U}\,r$

(19) **Right distributivity of $\mathcal{U}$ over $\Rightarrow$:** $(p \Rightarrow q)\,\mathcal{U}\,r \Rightarrow (p\,\mathcal{U}\,r \Rightarrow q\,\mathcal{U}\,r)$

(20) **Right zero of $\mathcal{U}$:** $p\,\mathcal{U}\,true \equiv true$

(21) **Left identity of $\mathcal{U}$:** $false\,\mathcal{U}\,q \equiv q$

(22) **Idempotency of $\mathcal{U}$:** $p\,\mathcal{U}\,p \equiv p$

(23) **$\mathcal{U}$ excluded middle:** $p\,\mathcal{U}\,q \vee p\,\mathcal{U}\,\neg q$

(24) $\neg p\,\mathcal{U}\,(q\,\mathcal{U}\,r) \wedge p\,\mathcal{U}\,r \Rightarrow q\,\mathcal{U}\,r$

(25) $p\,\mathcal{U}\,(\neg q\,\mathcal{U}\,r) \wedge q\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,r$

(26) $p\,\mathcal{U}\,q \wedge \neg q\,\mathcal{U}\,p \Rightarrow p$

(27) $p \wedge \neg p\,\mathcal{U}\,q \Rightarrow q$

(28) $p\,\mathcal{U}\,q \Rightarrow p \vee q$

(29) **$\mathcal{U}$ insertion:** $q \Rightarrow p\,\mathcal{U}\,q$

(30) $p \wedge q \Rightarrow p\,\mathcal{U}\,q$

(31) **Absorption:** $p \vee p\,\mathcal{U}\,q \equiv p \vee q$

(32) **Absorption:** $p\,\mathcal{U}\,q \vee q \equiv p\,\mathcal{U}\,q$

(33) **Absorption:** $p\,\mathcal{U}\,q \wedge q \equiv q$

(34) **Absorption:** $p\,\mathcal{U}\,q \vee (p \wedge q) \equiv p\,\mathcal{U}\,q$

(35) **Absorption:** $p\,\mathcal{U}\,q \wedge (p \vee q) \equiv p\,\mathcal{U}\,q$

(36) **Left absorption of $\mathcal{U}$:** $p\,\mathcal{U}\,(p\,\mathcal{U}\,q) \equiv p\,\mathcal{U}\,q$

(37) **Right absorption of $\mathcal{U}$:** $(p\,\mathcal{U}\,q)\,\mathcal{U}\,q \equiv p\,\mathcal{U}\,q$

Eventually $\Diamond$

(38) **Definition of $\Diamond$:** $\Diamond q \equiv true\, \mathcal{U}\, q$

(39) **Absorption of $\Diamond$ into $\mathcal{U}$:** $p\, \mathcal{U}\, q \wedge \Diamond q \equiv p\, \mathcal{U}\, q$

(40) **Absorption of $\mathcal{U}$ into $\Diamond$:** $p\, \mathcal{U}\, q \vee \Diamond q \equiv \Diamond q$

(41) **Absorption of $\mathcal{U}$ into $\Diamond$:** $p\, \mathcal{U} \Diamond q \equiv \Diamond q$

(42) **Eventuality:** $p\, \mathcal{U}\, q \Rightarrow \Diamond q$

(43) **Truth of $\Diamond$:** $\Diamond true \equiv true$

(44) **Falsehood of $\Diamond$:** $\Diamond false \equiv false$

(45) **Expansion of $\Diamond$:** $\Diamond p \equiv p \vee \bigcirc \Diamond p$

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

(47) **Weakening of $\Diamond$:** $\bigcirc p \Rightarrow \Diamond p$

(48) **Absorption of $\vee$ into $\Diamond$:** $p \vee \Diamond p \equiv \Diamond p$

(49) **Absorption of $\Diamond$ into $\wedge$:** $\Diamond p \wedge p \equiv p$

(50) **Absorption of $\Diamond$:** $\Diamond \Diamond p \equiv \Diamond p$

(51) **Exchange of $\bigcirc$ and $\Diamond$:** $\bigcirc \Diamond p \equiv \Diamond \bigcirc p$

(52) **Distributivity of $\Diamond$ over $\vee$:** $\Diamond (p \vee q) \equiv \Diamond p \vee \Diamond q$

(53) **Distributivity of $\Diamond$ over $\wedge$:** $\Diamond (p \wedge q) \Rightarrow \Diamond p \wedge \Diamond q$

Always $\Box$

(54) **Definition of $\Box$:** $\Box p \equiv \neg\Diamond\neg p$

(55) **Axiom, $\mathcal{U}$ Induction:** $\Box(p \Rightarrow (\bigcirc p \wedge q) \vee r) \Rightarrow (p \Rightarrow \Box q \vee q\,\mathcal{U}\,r)$

(56) **Axiom, $\mathcal{U}$ Induction:** $\Box(p \Rightarrow \bigcirc(p \vee q)) \Rightarrow (p \Rightarrow \Box p \vee p\,\mathcal{U}\,q)$

(57) **$\Box$ Induction:** $\Box(p \Rightarrow \bigcirc p) \Rightarrow (p \Rightarrow \Box p)$

(58) **$\Diamond$ Induction:** $\Box(\bigcirc p \Rightarrow p) \Rightarrow (\Diamond p \Rightarrow p)$

(59) $\Diamond p \equiv \neg\Box\neg p$

(60) **Dual of $\Box$:** $\neg\Box p \equiv \Diamond\neg p$

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

(62) **Dual of $\Diamond\Box$:** $\neg\Diamond\Box p \equiv \Box\Diamond\neg p$

(63) **Dual of $\Box\Diamond$:** $\neg\Box\Diamond p \equiv \Diamond\Box\neg p$

(64) **Truth of $\Box$:** $\Box true \equiv true$

(65) **Falsehood of $\Box$:** $\Box false \equiv false$

(66) **Expansion of $\Box$:** $\Box p \equiv p \wedge \bigcirc\Box p$

(67) **Expansion of $\Box$:** $\Box p \equiv p \wedge \bigcirc p \wedge \bigcirc\Box p$

(68) **Absorption of $\wedge$ into $\Box$:** $p \wedge \Box p \equiv \Box p$

(69) **Absorption of $\Box$ into $\vee$:** $\Box p \vee p \equiv p$

(70) **Absorption of $\Diamond$ into $\Box$:** $\Diamond p \wedge \Box p \equiv \Box p$

(71) **Absorption of $\Box$ into $\Diamond$:** $\Box p \vee \Diamond p \equiv \Diamond p$

(72) **Absorption of $\Box$:** $\Box\Box p \equiv \Box p$

(73) **Exchange of $\bigcirc$ and $\Box$:** $\bigcirc\Box p \equiv \Box\bigcirc p$

(74) $p \Rightarrow \Box p \equiv p \Rightarrow \bigcirc\Box p$

(75) $p \wedge \Diamond\neg p \Rightarrow \Diamond(p \wedge \bigcirc\neg p)$

(76) **Strengthening of $\Box$:** $\Box p \Rightarrow p$

(77) **Strengthening of $\Box$:** $\Box p \Rightarrow \Diamond p$

(78) **Strengthening of $\Box$:** $\Box p \Rightarrow \bigcirc p$

(79) **Strengthening of $\Box$:** $\Box p \Rightarrow \bigcirc\Box p$

(80) **$\bigcirc$ generalization:** $\Box p \Rightarrow \Box\bigcirc p$

(81) $\Box p \Rightarrow \neg(q\,\mathcal{U}\,\neg p)$

Temporal deduction

(82) **Temporal deduction:**

To prove $\Box P_1 \wedge \Box P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

Always, continued

(83) **Distributivity of $\wedge$ over $\mathcal{U}$:** $\Box p \wedge q\, \mathcal{U}\, r \Rightarrow (p \wedge q)\, \mathcal{U}\, (p \wedge r)$

(84) **$\mathcal{U}$ implication:** $\Box p \wedge \Diamond q \Rightarrow p\, \mathcal{U}\, q$

(85) **Right monotonicity of $\mathcal{U}$:** $\Box (p \Rightarrow q) \Rightarrow (r\, \mathcal{U}\, p \Rightarrow r\, \mathcal{U}\, q)$

(86) **Left monotonicity of $\mathcal{U}$:** $\Box (p \Rightarrow q) \Rightarrow (p\, \mathcal{U}\, r \Rightarrow q\, \mathcal{U}\, r)$

(87) **Distributivity of $\neg$ over $\Box$:** $\Box \neg p \Rightarrow \neg \Box p$

(88) **Distributivity of $\Diamond$ over $\wedge$:** $\Box p \wedge \Diamond q \Rightarrow \Diamond (p \wedge q)$

(89) **$\Diamond$ excluded middle:** $\Diamond p \vee \Box \neg p$

(90) **$\Box$ excluded middle:** $\Box p \vee \Diamond \neg p$

(91) **Temporal excluded middle:** $\Diamond p \vee \Diamond \neg p$

(92) **$\Diamond$ contradiction:** $\Diamond p \wedge \Box \neg p \equiv false$

(93) **$\Box$ contradiction:** $\Box p \wedge \Diamond \neg p \equiv false$

(94) **Temporal contradiction:** $\Box p \wedge \Box \neg p \equiv false$

(95) **$\Box \Diamond$ excluded middle:** $\Box \Diamond p \vee \Diamond \Box \neg p$

(96) **$\Diamond \Box$ excluded middle:** $\Diamond \Box p \vee \Box \Diamond \neg p$

(97) **$\Box \Diamond$ contradiction:** $\Box \Diamond p \wedge \Diamond \Box \neg p \equiv false$

(98) **$\Diamond \Box$ contradiction:** $\Diamond \Box p \wedge \Box \Diamond \neg p \equiv false$

(99) **Distributivity of $\Box$ over $\wedge$:** $\Box(p \wedge q) \equiv \Box p \wedge \Box q$

(100) **Distributivity of $\Box$ over $\vee$:** $\Box p \vee \Box q \Rightarrow \Box(p \vee q)$

(101) **Logical equivalence law of $\bigcirc$:** $\Box(p \equiv q) \Rightarrow (\bigcirc p \equiv \bigcirc q)$

(102) **Logical equivalence law of $\Diamond$:** $\Box(p \equiv q) \Rightarrow (\Diamond p \equiv \Diamond q)$

(103) **Logical equivalence law of $\Box$:** $\Box(p \equiv q) \Rightarrow (\Box p \equiv \Box q)$

(104) **Distributivity of $\Diamond$ over $\Rightarrow$:** $\Diamond(p \Rightarrow q) \equiv (\Box p \Rightarrow \Diamond q)$

(105) **Distributivity of $\Diamond$ over $\Rightarrow$:** $(\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond(p \Rightarrow q)$

(106) $\wedge$ **frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p \wedge q))$

(107) $\wedge$ **frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \wedge q))$

(108) $\wedge$ **frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p \wedge q))$

(109) $\vee$ **frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p \vee q))$

(110) $\vee$ **frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \vee q))$

(111) $\vee$ **frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p \vee q))$

(112) $\Rightarrow$ **frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p \Rightarrow q))$

(113) $\Rightarrow$ **frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \Rightarrow q))$

(114) $\Rightarrow$ **frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p \Rightarrow q))$

(115) $\equiv$ **frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p \equiv q))$

(116) $\equiv$ **frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p \equiv q))$

(117) $\equiv$ **frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p \equiv q))$

(118) **Monotonicity of $\bigcirc$:** $\Box(p \Rightarrow q) \Rightarrow (\bigcirc p \Rightarrow \bigcirc q)$

(119) **Monotonicity of $\Diamond$:** $\Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$

(120) **Monotonicity of $\Box$:** $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

(121) **Consequence rule of $\circ$:** $\Box((p \Rightarrow q) \wedge (q \Rightarrow \circ r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \circ s)$

(122) **Consequence rule of $\Diamond$:** $\Box((p \Rightarrow q) \wedge (q \Rightarrow \Diamond r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \Diamond s)$

(123) **Consequence rule of $\Box$:** $\Box((p \Rightarrow q) \wedge (q \Rightarrow \Box r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \Box s)$

(124) **Catenation rule of $\Diamond$:** $\Box((p \Rightarrow \Diamond q) \wedge (q \Rightarrow \Diamond r)) \Rightarrow (p \Rightarrow \Diamond r)$

(125) **Catenation rule of $\Box$:** $\Box((p \Rightarrow \Box q) \wedge (q \Rightarrow \Box r)) \Rightarrow (p \Rightarrow \Box r)$

(126) **Catenation rule of $\mathcal{U}$:** $\Box((p \Rightarrow q \,\mathcal{U}\, r) \wedge (r \Rightarrow q \,\mathcal{U}\, s)) \Rightarrow (p \Rightarrow q \,\mathcal{U}\, s)$

(127) **$\mathcal{U}$ strengthening rule:** $\Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p \,\mathcal{U}\, q \Rightarrow r \,\mathcal{U}\, s)$

(128) **Induction rule $\Diamond$:** $\Box(p \vee \circ q \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow q)$

(129) **Induction rule $\Box$:** $\Box(p \Rightarrow q \wedge \circ p) \Rightarrow (p \Rightarrow \Box q)$

(130) **Induction rule $\mathcal{U}$:** $\Box(p \Rightarrow \neg q \wedge \circ p) \Rightarrow (p \Rightarrow \neg(r \,\mathcal{U}\, q))$

(131) **$\Diamond$ Confluence:** $\Box((p \Rightarrow \Diamond(q \vee r)) \wedge (q \Rightarrow \Diamond t) \wedge (r \Rightarrow \Diamond t)) \Rightarrow (p \Rightarrow \Diamond t)$

(132) **Temporal generalization law:** $\Box(\Box p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$

(133) **Temporal particularization law:** $\Box(p \Rightarrow \Diamond q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$

(134) $\Box(p \Rightarrow \circ q) \Rightarrow (p \Rightarrow \Diamond q)$

(135) $\Box(p \Rightarrow \circ \neg p) \Rightarrow (p \Rightarrow \neg \Box p)$

Proof metatheorems

(136) **Metatheorem:** P is a theorem iff $\Box P$ is a theorem.

(137) **Metatheorem $\circ$:** If $P \Rightarrow Q$ is a theorem then $\circ P \Rightarrow \circ Q$ is a theorem.

(138) **Metatheorem $\Diamond$:** If $P \Rightarrow Q$ is a theorem then $\Diamond P \Rightarrow \Diamond Q$ is a theorem.

(139) **Metatheorem $\Box$:** If $P \Rightarrow Q$ is a theorem then $\Box P \Rightarrow \Box Q$ is a theorem.

Always, continued

(140) $\mathcal{U}\,\Box$ **implication:** $p\,\mathcal{U}\,\Box q \Rightarrow \Box(p\,\mathcal{U}\,q)$

(141) **Absorption of $\mathcal{U}$ into $\Box$:** $p\,\mathcal{U}\,\Box p \equiv \Box p$

(142) **Right $\wedge\,\mathcal{U}$ strengthening:** $p\,\mathcal{U}\,(q \wedge r) \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{U}\,r)$

(143) **Left $\wedge\,\mathcal{U}$ strengthening:** $(p \wedge q)\,\mathcal{U}\,r \Rightarrow (p\,\mathcal{U}\,q)\,\mathcal{U}\,r$

(144) **Left $\wedge\,\mathcal{U}$ ordering:** $(p \wedge q)\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{U}\,r)$

(145) $\Diamond\Box$ **implication:** $\Diamond\Box p \Rightarrow \Box\Diamond p$

(146) $\Box\Diamond$ **excluded middle:** $\Box\Diamond p \vee \Box\Diamond\neg p$

(147) $\Diamond\Box$ **contradiction:** $\Diamond\Box p \wedge \Diamond\Box\neg p \;\equiv\; false$

(148) $\mathcal{U}$ **frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p\,\mathcal{U}\,q))$

(149) $\mathcal{U}$ **frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p\,\mathcal{U}\,q))$

(150) $\mathcal{U}$ **frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p\,\mathcal{U}\,q))$

(151) **Absorption of $\Diamond$ into $\Box\Diamond$:** $\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(152) **Absorption of $\Box$ into $\Diamond\Box$:** $\Box\Diamond\Box p \equiv \Diamond\Box p$

(153) **Absorption of $\Box\Diamond$:** $\Box\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(154) **Absorption of $\Diamond\Box$:** $\Diamond\Box\Diamond\Box p \equiv \Diamond\Box p$

(155) **Absorption of $\bigcirc$ into $\Box\Diamond$:** $\bigcirc\Box\Diamond p \equiv \Box\Diamond p$

(156) **Absorption of $\bigcirc$ into $\Diamond\Box$:** $\bigcirc\Diamond\Box p \equiv \Diamond\Box p$

(157) **Monotonicity of $\Box\Diamond$:** $\Box(p \Rightarrow q) \Rightarrow (\Box\Diamond p \Rightarrow \Box\Diamond q)$

(158) **Monotonicity of $\Diamond\Box$:** $\Box(p \Rightarrow q) \Rightarrow (\Diamond\Box p \Rightarrow \Diamond\Box q)$

(159) **Distributivity of $\Box\Diamond$ over $\wedge$:** $\Box\Diamond(p \wedge q) \Rightarrow \Box\Diamond p \wedge \Box\Diamond q$

(160) **Distributivity of $\Diamond\Box$ over $\vee$:** $\Diamond\Box p \vee \Diamond\Box q \Rightarrow \Diamond\Box(p \vee q)$

(161) **Distributivity of $\Box\Diamond$ over $\vee$:** $\Box\Diamond(p \vee q) \equiv \Box\Diamond p \vee \Box\Diamond q$

(162) **Distributivity of $\Diamond\Box$ over $\wedge$:** $\Diamond\Box(p \wedge q) \equiv \Diamond\Box p \wedge \Diamond\Box q$

(163) **Eventual latching:** $\Diamond\Box(p \Rightarrow \Box q) \equiv \Diamond\Box\neg p \vee \Diamond\Box q$

(164) $\Box(\Box\Diamond p \Rightarrow \Diamond q) \equiv \Diamond\Box\neg p \vee \Box\Diamond q$

(165) $\Box((p \vee \Box q) \wedge (\Box p \vee q)) \equiv \Box p \vee \Box q$

(166) $\Diamond\Box p \wedge \Box\Diamond q \Rightarrow \Box\Diamond(p \wedge q)$

(167) $\Box((\Box p \Rightarrow \Diamond q) \wedge (q \Rightarrow \bigcirc r)) \Rightarrow (\Box p \Rightarrow \bigcirc\Box\Diamond r)$

(168) **Progress proof rule:** $\Diamond\Box p \wedge \Box(\Box p \Rightarrow \Diamond q) \Rightarrow \Diamond q$

Wait $\mathcal{W}$

(169) **Definition of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv \Box p \vee p \,\mathcal{U}\, q$

(170) **Axiom, Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p \,\mathcal{W}\, q) \equiv \neg q \,\mathcal{U}\, (\neg p \wedge \neg q)$

(171) **$\mathcal{U}$ in terms of $\mathcal{W}$:** $p \,\mathcal{U}\, q \equiv p \,\mathcal{W}\, q \wedge \Diamond q$

(172) $p \,\mathcal{W}\, q \equiv \Box (p \wedge \neg q) \vee p \,\mathcal{U}\, q$

(173) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p \,\mathcal{U}\, q) \equiv \neg q \,\mathcal{W}\, (\neg p \wedge \neg q)$

(174) **$\mathcal{U}$ implication:** $p \,\mathcal{U}\, q \Rightarrow p \,\mathcal{W}\, q$

(175) **Distributivity of $\wedge$ over $\mathcal{W}$:** $\Box p \wedge q \,\mathcal{W}\, r \Rightarrow (p \wedge q) \,\mathcal{W}\, (p \wedge r)$

(176) **$\mathcal{W} \Diamond$ equivalence:** $p \,\mathcal{W} \Diamond q \equiv \Box p \vee \Diamond q$

(177) **$\mathcal{W} \Box$ implication:** $p \,\mathcal{W} \Box q \Rightarrow \Box (p \,\mathcal{W}\, q)$

(178) **Absorption of $\mathcal{W}$ into $\Box$:** $p \,\mathcal{W} \Box p \equiv \Box p$

(179) **Perpetuity:** $\Box p \Rightarrow p \,\mathcal{W}\, q$

(180) **Distributivity of $\circ$ over $\mathcal{W}$:** $\circ (p \,\mathcal{W}\, q) \equiv \circ p \,\mathcal{W} \circ q$

(181) **Expansion of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv q \vee (p \wedge \circ (p \,\mathcal{W}\, q))$

(182) **$\mathcal{W}$ excluded middle:** $p \,\mathcal{W}\, q \vee p \,\mathcal{W} \neg q$

(183) **Left zero of $\mathcal{W}$:** $true \,\mathcal{W}\, q \equiv true$

(184) **Left distributivity of $\mathcal{W}$ over $\vee$:** $p \,\mathcal{W}\, (q \vee r) \equiv p \,\mathcal{W}\, q \vee p \,\mathcal{W}\, r$

(185) **Right distributivity of $\mathcal{W}$ over $\vee$:** $p \,\mathcal{W}\, r \vee q \,\mathcal{W}\, r \Rightarrow (p \vee q) \,\mathcal{W}\, r$

(186) **Left distributivity of $\mathcal{W}$ over $\wedge$:** $p \,\mathcal{W}\, (q \wedge r) \Rightarrow p \,\mathcal{W}\, q \wedge p \,\mathcal{W}\, r$

(187) **Right distributivity of $\mathcal{W}$ over $\wedge$:** $(p \wedge q) \,\mathcal{W}\, r \equiv p \,\mathcal{W}\, r \wedge q \,\mathcal{W}\, r$

(188) **Right distributivity of $\mathcal{W}$ over $\Rightarrow$:** $(p \Rightarrow q) \,\mathcal{W}\, r \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r)$

(189) **Disjunction rule of $\mathcal{W}$:** $p \, \mathcal{W} \, q \equiv (p \vee q) \, \mathcal{W} \, q$

(190) **Disjunction rule of $\mathcal{U}$:** $p \, \mathcal{U} \, q \equiv (p \vee q) \, \mathcal{U} \, q$

(191) **Rule of $\mathcal{W}$:** $\neg q \, \mathcal{W} \, q$

(192) **Rule of $\mathcal{U}$:** $\neg q \, \mathcal{U} \, q \; \equiv \; \Diamond q$

(193) $(p \Rightarrow q) \, \mathcal{W} \, p$

(194) $\Diamond p \Rightarrow (p \Rightarrow q) \, \mathcal{U} \, p$

(195) **Conjunction rule of $\mathcal{W}$:** $p \, \mathcal{W} \, q \equiv (p \wedge \neg q) \, \mathcal{W} \, q$

(196) **Conjunction rule of $\mathcal{U}$:** $p \, \mathcal{U} \, q \equiv (p \wedge \neg q) \, \mathcal{U} \, q$

(197) **Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p \, \mathcal{W} \, q) \equiv (p \wedge \neg q) \, \mathcal{U} \, (\neg p \wedge \neg q)$

(198) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p \, \mathcal{U} \, q) \equiv (p \wedge \neg q) \, \mathcal{W} \, (\neg p \wedge \neg q)$

(199) **Dual of $\mathcal{U}$:** $\neg(\neg p \, \mathcal{U} \, \neg q) \equiv q \, \mathcal{W} \, (p \wedge q)$

(200) **Dual of $\mathcal{U}$:** $\neg(\neg p \, \mathcal{U} \, \neg q) \equiv (\neg p \wedge q) \, \mathcal{W} \, (p \wedge q)$

(201) **Dual of $\mathcal{W}$:** $\neg(\neg p \, \mathcal{W} \, \neg q) \equiv q \, \mathcal{U} \, (p \wedge q)$

(202) **Dual of $\mathcal{W}$:** $\neg(\neg p \, \mathcal{W} \, \neg q) \equiv (\neg p \wedge q) \, \mathcal{U} \, (p \wedge q)$

(203) **Idempotency of $\mathcal{W}$:** $p \, \mathcal{W} \, p \equiv p$

(204) **Right zero of $\mathcal{W}$:** $p \, \mathcal{W} \, true \equiv true$

(205) **Left identity of $\mathcal{W}$:** $false \, \mathcal{W} \, q \equiv q$

(206) $p \, \mathcal{W} \, q \Rightarrow p \vee q$

(207) $\Box (p \vee q) \Rightarrow p \, \mathcal{W} \, q$

(208) $\Box (\neg q \Rightarrow p) \Rightarrow p \, \mathcal{W} \, q$

(209) $\mathcal{W}$ **insertion:** $q \Rightarrow p \,\mathcal{W}\, q$

(210) $\mathcal{W}$ **frame law of** $\bigcirc$ **:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc (p \,\mathcal{W}\, q))$

(211) $\mathcal{W}$ **frame law of** $\Diamond$ **:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond (p \,\mathcal{W}\, q))$

(212) $\mathcal{W}$ **frame law of** $\Box$ **:** $\Box p \Rightarrow (\Box q \Rightarrow \Box (p \,\mathcal{W}\, q))$

(213) $\mathcal{W}$ **induction:** $\Box (p \Rightarrow (\bigcirc p \wedge q) \vee r) \Rightarrow (p \Rightarrow q \,\mathcal{W}\, r)$

(214) $\mathcal{W}$ **induction:** $\Box (p \Rightarrow \bigcirc (p \vee q)) \Rightarrow (p \Rightarrow p \,\mathcal{W}\, q)$

(215) $\mathcal{W}$ **induction:** $\Box (p \Rightarrow \bigcirc p) \Rightarrow (p \Rightarrow p \,\mathcal{W}\, q)$

(216) $\mathcal{W}$ **induction:** $\Box (p \Rightarrow q \wedge \bigcirc p) \Rightarrow (p \Rightarrow p \,\mathcal{W}\, q)$

(217) **Absorption:** $p \vee p \,\mathcal{W}\, q \equiv p \vee q$

(218) **Absorption:** $p \,\mathcal{W}\, q \vee q \equiv p \,\mathcal{W}\, q$

(219) **Absorption:** $p \,\mathcal{W}\, q \wedge q \equiv q$

(220) **Absorption:** $p \,\mathcal{W}\, q \wedge (p \vee q) \equiv p \,\mathcal{W}\, q$

(221) **Absorption:** $p \,\mathcal{W}\, q \vee (p \wedge q) \equiv p \,\mathcal{W}\, q$

(222) **Left absorption of** $\mathcal{W}$ **:** $p \,\mathcal{W}\, (p \,\mathcal{W}\, q) \equiv p \,\mathcal{W}\, q$

(223) **Right absorption of** $\mathcal{W}$ **:** $(p \,\mathcal{W}\, q) \,\mathcal{W}\, q \equiv p \,\mathcal{W}\, q$

(224) $\Box$ **to** $\mathcal{W}$ **law:** $\Box p \equiv p \,\mathcal{W}\, false$

(225) $\Diamond$ **to** $\mathcal{W}$ **law:** $\Diamond p \equiv \neg(\neg p \,\mathcal{W}\, false)$

(226) $\mathcal{W}$ **implication:** $p \,\mathcal{W}\, q \Rightarrow \Box p \vee \Diamond q$

(227) **Absorption:** $p \,\mathcal{W}\, (p \,\mathcal{U}\, q) \equiv p \,\mathcal{W}\, q$

(228) **Absorption:** $(p \,\mathcal{U}\, q) \,\mathcal{W}\, q \equiv p \,\mathcal{U}\, q$

(229) **Absorption:** $p\,\mathcal{U}\,(p\,\mathcal{W}\,q) \equiv p\,\mathcal{W}\,q$

(230) **Absorption:** $(p\,\mathcal{W}\,q)\,\mathcal{U}\,q \equiv p\,\mathcal{U}\,q$

(231) **Absorption of $\mathcal{W}$ into $\Diamond$:** $\Diamond q\,\mathcal{W}\,q \equiv \Diamond q$

(232) **Absorption of $\mathcal{W}$ into $\Box$:** $\Box p \wedge p\,\mathcal{W}\,q \equiv \Box p$

(233) **Absorption of $\Box$ into $\mathcal{W}$:** $\Box p \vee p\,\mathcal{W}\,q \equiv p\,\mathcal{W}\,q$

(234) $p\,\mathcal{W}\,q \wedge \Box\neg q \Rightarrow \Box p$

(235) $\Box p \Rightarrow p\,\mathcal{U}\,q \vee \Box\neg q$

(236) $\neg\Box p \wedge p\,\mathcal{W}\,q \Rightarrow \Diamond q$

(237) $\Diamond q \Rightarrow \neg\Box p \vee p\,\mathcal{U}\,q$

(238) **Left monotonicity of $\mathcal{W}$:** $\Box(p \Rightarrow q) \Rightarrow (p\,\mathcal{W}\,r \Rightarrow q\,\mathcal{W}\,r)$

(239) **Right monotonicity of $\mathcal{W}$:** $\Box(p \Rightarrow q) \Rightarrow (r\,\mathcal{W}\,p \Rightarrow r\,\mathcal{W}\,q)$

(240) **$\mathcal{W}$ strengthening rule:** $\Box((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p\,\mathcal{W}\,q \Rightarrow r\,\mathcal{W}\,s)$

(241) **$\mathcal{W}$ catenation rule:** $\Box((p \Rightarrow q\,\mathcal{W}\,r) \wedge (r \Rightarrow q\,\mathcal{W}\,s)) \Rightarrow (p \Rightarrow q\,\mathcal{W}\,s)$

(242) **Left $\mathcal{U}$ $\mathcal{W}$ implication:** $(p\,\mathcal{U}\,q)\,\mathcal{W}\,r \Rightarrow (p\,\mathcal{W}\,q)\,\mathcal{W}\,r$

(243) **Right $\mathcal{W}$ $\mathcal{U}$ implication:** $p\,\mathcal{W}\,(q\,\mathcal{U}\,r) \Rightarrow p\,\mathcal{W}\,(q\,\mathcal{W}\,r)$

(244) **Right $\mathcal{U}$ $\mathcal{U}$ implication:** $p\,\mathcal{U}\,(q\,\mathcal{U}\,r) \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{W}\,r)$

(245) **Left $\mathcal{U}$ $\mathcal{U}$ implication:** $(p\,\mathcal{U}\,q)\,\mathcal{U}\,r \Rightarrow (p\,\mathcal{W}\,q)\,\mathcal{U}\,r$

(246) **Left $\mathcal{U}\vee$ strengthening:** $(p\,\mathcal{U}\,q)\,\mathcal{U}\,r \Rightarrow (p \vee q)\,\mathcal{U}\,r$

(247) **Left $\mathcal{W}\vee$ strengthening:** $(p\,\mathcal{W}\,q)\,\mathcal{W}\,r \Rightarrow (p \vee q)\,\mathcal{W}\,r$

(248) **Right $\mathcal{W}\vee$ strengthening:** $p\,\mathcal{W}\,(q\,\mathcal{W}\,r) \Rightarrow p\,\mathcal{W}\,(q \vee r)$

(249) **Right $\mathcal{W}\vee$ ordering:** $p\,\mathcal{W}\,(q\,\mathcal{W}\,r) \Rightarrow (p \vee q)\,\mathcal{W}\,r$

(250) **Right $\wedge\,\mathcal{W}$ ordering:** $p\,\mathcal{W}\,(q \wedge r) \Rightarrow (p\,\mathcal{W}\,q)\,\mathcal{W}\,r$

(251) **$\mathcal{U}$ ordering:** $\neg p\,\mathcal{U}\,q \vee \neg q\,\mathcal{U}\,p \equiv \Diamond(p \vee q)$

(252) **$\mathcal{W}$ ordering:** $\neg p\,\mathcal{W}\,q \vee \neg q\,\mathcal{W}\,p$

(253) **$\mathcal{W}$ implication ordering:** $p\,\mathcal{W}\,q \wedge \neg q\,\mathcal{W}\,r \Rightarrow p\,\mathcal{W}\,r$

(254) **Lemmon formula:** $\Box(\Box p \Rightarrow q) \vee \Box(\Box q \Rightarrow p)$

Additional Theorems of Linear Temporal Logic

Temporal Modus Ponens

(S1) **Negation of $\Rightarrow$:** $\neg(p \Rightarrow q) \equiv p \wedge \neg q$

(S2) **Temporal Modus Ponens of $\bigcirc$:** $\Box\,(p \Rightarrow q) \wedge \bigcirc\, p \Rightarrow \bigcirc\, q$

(S3) **Temporal Modus Ponens of $\Diamond$:** $\Box\,(p \Rightarrow q) \wedge \Diamond\, p \Rightarrow \Diamond\, q$

(S4) **Temporal Modus Ponens of $\Box$:** $\Box\,(p \Rightarrow q) \wedge \Box\, p \Rightarrow \Box\, q$

(S5) **Temporal Modus Ponens of $\Box\Diamond$:** $\Box\,(p \Rightarrow q) \wedge \Box\Diamond\, p \Rightarrow \Box\Diamond\, q$

(S6) **Temporal Modus Ponens of $\Diamond\Box$:** $\Box\,(p \Rightarrow q) \wedge \Diamond\Box\, p \Rightarrow \Diamond\Box\, q$

(S7) **Prior Formula:** $\neg\Diamond\, q \Rightarrow (\Box\,(p \Rightarrow q) \Rightarrow \neg\Diamond\, p)$

Spot

Theorems from Spot are distributed throughout the paper.

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(S8) $\Box\, p \Rightarrow \Diamond\, q \equiv p\, \mathcal{U}\, (q \vee \neg p)$

(S9) $\Diamond\,(p \Rightarrow q) \equiv p\, \mathcal{U}\, (q \vee \neg p)$

(S10) $p\, \mathcal{U}\, \neg p \equiv \neg\Box\, p$

(S11) $p\, \mathcal{U}\, (q \vee \neg p) \equiv \Box\, p \Rightarrow p\, \mathcal{U}\, q$

(S12) $\Box\, p \Rightarrow (p\, \mathcal{U}\, q)\, \mathcal{U}\, p$

(S13) $p\, \mathcal{U}\, (q\, \mathcal{U}\, p) \equiv q\, \mathcal{U}\, p$

(S14) $(p\, \mathcal{U}\, q)\, \mathcal{U}\, p \equiv q\, \mathcal{U}\, p$

Induction

(S15) **(165) Lemma F:** $\Box p \vee \Box q \Rightarrow \bigcirc(\Box p \vee \Box q)$

(S16) $\Box p \wedge \Box q \Rightarrow \bigcirc(\Box p \wedge \Box q)$

(S17) $\bigcirc(\Diamond p \vee \Diamond q) \Rightarrow (\Diamond p \vee \Diamond q)$

(S18) $\bigcirc(\Diamond p \wedge \Diamond q) \Rightarrow (\Diamond p \wedge \Diamond q)$

(S19) $\Box(p \Rightarrow \bigcirc p) \Rightarrow \Box(p \Rightarrow \Box p)$

(S20) $\Box p \Rightarrow (p \Rightarrow \Box p)$

(S21) $\Diamond \Box p \Rightarrow \Diamond(p \Rightarrow \Box p)$

(S22) $\Box p \Rightarrow \Diamond(p \Rightarrow \Box p)$

(S23) **$\Box\Diamond$ induction:** $\Box((p \Rightarrow q \wedge \bigcirc p) \wedge (q \Rightarrow \Diamond p)) \Rightarrow (p \Rightarrow \Box\Diamond p)$

(S24) **$\Box\Diamond$ induction:** $\Box((p \Rightarrow \bigcirc p) \wedge (p \Rightarrow \Diamond p)) \Rightarrow (p \Rightarrow \Box\Diamond p)$

(S25) **Obligation induction:** $\Box(p \Rightarrow \bigcirc p) \vee \Box(p \Rightarrow \bigcirc q) \Rightarrow (p \Rightarrow p\, \mathcal{W} \Diamond q)$

(S26) **Induction rule $\mathcal{U}$:** $\Box(p \Rightarrow \bigcirc(p \wedge q)) \Rightarrow (p \Rightarrow p\, \mathcal{U}\, q)$

(S27) **Dummett induction:** $\Box(p \equiv \bigcirc p) \Rightarrow (\Diamond\Box p \Rightarrow \Box p)$

Absorption

(S28) **Absorption of $\Diamond$ into $\neg\Box$:** $\Diamond\neg\Box p \equiv \neg\Box p$

(S29) **Absorption of $\Box$ into $\neg\Diamond$:** $\Box\neg\Diamond p \equiv \neg\Diamond p$

(S30) $p\, \mathcal{W}\, q \vee \neg q\, \mathcal{W}\, p \;\equiv\; q \vee \neg q\, \mathcal{W}\, p$

Duality

(S31) **Definition of Weak Release $\mathcal{R}$:** $p\,\mathcal{R}\,q \equiv q\,\mathcal{W}\,(p \wedge q)$

(S32) **Definition of Strong Release $\mathcal{M}$:** $p\,\mathcal{M}\,q \equiv q\,\mathcal{U}\,(p \wedge q)$

(S33) $\neg(q\,\mathcal{U}\,(\neg p \wedge q)) \vee \neg(p\,\mathcal{U}\,(\neg q \wedge p))$

(S34) $\Diamond((p \wedge \Diamond q) \vee (\Diamond p \wedge q)) \equiv \Diamond p \wedge \Diamond q$

(S35) $\Diamond p \wedge \Diamond q \Rightarrow (\Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(\Diamond p \wedge q))$

(S36) $p \not\Leftarrow q \;\equiv\; \neg(p \Leftarrow q)$

(S37) $p \Leftarrow q \;\equiv\; \neg q \vee p$

(S38) $p \not\Leftarrow q \;\equiv\; \neg(q \Rightarrow p)$

Next $\circ$

(S39) $\Box(p \Rightarrow \circ\, p) \Rightarrow \Box(p \Rightarrow \circ\,(p \vee q))$

(S40) $\Box(p \Rightarrow \circ\,(p \Rightarrow q) \wedge \circ\, p) \equiv \Box(p \Rightarrow \circ\,(p \wedge q))$

(S41) $p \wedge \circ\, p \equiv \neg(\circ\, p \Rightarrow \neg p)$

Until $\mathcal{U}$

(S42) $\Box(p \wedge q) \Rightarrow p\,\mathcal{U}\,q$

(S43) $p\,\mathcal{U}\,(q \vee \Diamond r) \;\equiv\; p\,\mathcal{U}\,q \vee \Diamond r$

(S44) $\Box p\,\mathcal{U}\,q \Rightarrow p\,\mathcal{U}\,q \vee \Box\neg q$

(S45) $\neg(p\,\mathcal{U}\,q) \Rightarrow p\,\mathcal{U}\,\neg q$

(S46) **Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p\,\mathcal{W}\,q) \equiv \neg q\,\mathcal{W}\,(\neg p \wedge \neg q) \wedge \neg\Box p$

(S47) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p\,\mathcal{U}\,q) \equiv \neg q\,\mathcal{U}\,(\neg p \wedge \neg q) \vee \Box\neg q$

(S48) **Weak symmetry of $\mathcal{U}$:** $(p \vee q)\,\mathcal{U}\,q \Rightarrow q\,\mathcal{U}\,(p \vee q)$

(S49) **Generalized $\mathcal{U}$ excluded middle:** $p\,\mathcal{U}\,q \vee r\,\mathcal{U}\,\neg q$

(S50) $((\Box p \vee \Diamond q) \equiv p\,\mathcal{W}\,\Diamond q) \equiv \Box p \vee \Diamond q \vee \Box\neg q\,\mathcal{U}\,(\neg p \wedge \Box\neg q)$

In-state expansion of $\mathcal{U}$

(S51) **In-state next-state equivalence:** $q \vee (p \wedge p \,\mathcal{U}\, q) \equiv q \vee (p \wedge \bigcirc (p \,\mathcal{U}\, q))$

(S52) **In-state expansion of $\mathcal{U}$:** $p \,\mathcal{U}\, q \equiv q \vee (p \wedge p \,\mathcal{U}\, q)$

Nested insertion

(S53) **Nested insertion:** $r \Rightarrow p \,\mathcal{U}\, (q \,\mathcal{U}\, r)$

(S54) **Nested insertion:** $r \Rightarrow (p \,\mathcal{U}\, q) \,\mathcal{U}\, r$

(S55) **Indefinite nested insertion:** $x_n \Rightarrow x_1 \,\mathcal{U}\, (x_2 \,\mathcal{U}\, (\ldots \,\mathcal{U}\, (x_{n-1} \,\mathcal{U}\, x_n \underbrace{)\ldots))}_{n-2\ times}$ for $n \geq 3$

(S56) **Indefinite nested insertion:** $x_n \Rightarrow \underbrace{(\ldots(}_{n-2\ times} x_1 \,\mathcal{U}\, x_2) \,\mathcal{U}\, x_3) \ldots \,\mathcal{U}\, x_{n-1}) \,\mathcal{U}\, x_n$

Eventually $\Diamond$

(S57) $\Diamond p \wedge \Diamond q \Rightarrow \Diamond (p \wedge \Diamond q) \vee \Diamond (\Diamond p \wedge q)$

(S58) $\Box p \wedge \Diamond q \Rightarrow \Diamond (\Box p \wedge q)$

(S59) $\Box (p \Rightarrow \bigcirc (p \Rightarrow q)) \Rightarrow \Diamond (p \Rightarrow q)$

(S60) $\Diamond (p \,\mathcal{U}\, q) \;\equiv\; \Diamond q$

(S61) $p \,\mathcal{U}\, \Diamond q \;\equiv\; \Diamond (p \,\mathcal{U}\, q)$

(S62) $\Diamond q \Rightarrow (p \,\mathcal{W}\, q \;\equiv\; p \,\mathcal{U}\, q)$

Always $\Box$

(S63) $\Box (p \vee q) \wedge \Box (\Box p \vee q) \wedge \Box (p \vee \Box q) \Rightarrow \Box p \vee \Box q$

(S64) $\Box (\Box p \vee q) \wedge \Box (p \vee \Box q) \Rightarrow \Box p \vee \Box q$

(S65) $\Box (p \wedge \Box p \Rightarrow q) \vee \Box (q \wedge \Box q \Rightarrow p)$

(S66) $\Box (\Box p \Rightarrow \Box q) \vee \Box (\Box q \Rightarrow \Box p)$

(S67) $\Box ((p \Rightarrow \Box p) \Rightarrow \Box p) \;\equiv\; \Box p$

Always Eventually $\Box\Diamond$ and its Dual $\Diamond\Box$

(S68) $\Box(p \Rightarrow \bigcirc(p \Rightarrow q)) \Rightarrow \Box\Diamond(p \Rightarrow q)$

(S69) $\Box\Diamond(p \vee \bigcirc q) \equiv \Box\Diamond(p \vee q)$

(S70) $\Diamond\Box(p \wedge \bigcirc q) \equiv \Diamond\Box(p \wedge q)$

(S71) $\Box\Diamond(p \vee \Diamond q) \equiv \Box\Diamond(p \vee q)$

(S72) $\Diamond\Box(p \wedge \Box q) \equiv \Diamond\Box(p \wedge q)$

(S73) $\bigcirc p \vee \Box\Diamond q \equiv \bigcirc(p \vee \Box\Diamond q)$

(S74) $\bigcirc p \wedge \Diamond\Box q \equiv \bigcirc(p \wedge \Diamond\Box q)$

Wait $\mathcal{W}$

(S75) **In-state expansion of $\mathcal{W}$:** $p\,\mathcal{W}\,q \equiv q \vee (p \wedge p\,\mathcal{W}\,q)$

(S76) $\neg(p\,\mathcal{W}\,q) \Rightarrow p\,\mathcal{W}\,\neg q$

(S77) $p\,\mathcal{W}\,\Box q \wedge \Diamond\Box q \Rightarrow \Box(p\,\mathcal{W}\,q)$

(S78) **Generalized $\mathcal{W}$ excluded middle:** $p\,\mathcal{W}\,q \vee r\,\mathcal{W}\,\neg q$

(S79) $p\,\mathcal{W}\,q \equiv \Diamond\neg p \Rightarrow p\,\mathcal{U}\,q$

(S80) $p\,\mathcal{W}\,q \equiv p\,\mathcal{U}\,(q \vee \Box p)$

(S81) $q\,\mathcal{W}\,\Box\neg p \Rightarrow (\Diamond p \Rightarrow q)$

(S82) $q\,\mathcal{W}\,\Box\neg q \Rightarrow (\Box(\bigcirc q \Rightarrow q) \Rightarrow (\Diamond q \Rightarrow q))$

(S83) $p\,\mathcal{W}\,\Box p \equiv \Box p$

(S84) $\Box p\,\mathcal{W}\,q \equiv \Box p \vee q$

(S85) $\Box(p\,\mathcal{W}\,q) \equiv \Box(p \vee q)$

(S86) $p \mathcal{W} q \equiv p \mathcal{W} (q \vee \Box p)$

(S87) $p \mathcal{W} (q \vee \Diamond r) \;\equiv\; p \mathcal{W} q \vee \Diamond r$

(S88) $p \mathcal{U} r \wedge q \mathcal{W} r \;\equiv\; (p \wedge q) \mathcal{U} r$

(S89) $p \mathcal{U} q \vee p \mathcal{W} r \;\equiv\; p \mathcal{W} (q \vee r)$

(S90) $p \mathcal{W} q \vee \Diamond q \;\equiv\; \Box p \vee \Diamond q$

(S91) $p \mathcal{W} \Diamond q \;\equiv\; p \mathcal{W} q \vee \Diamond q$

(S92) $p \mathcal{W} q \vee q \mathcal{W} p \;\equiv\; p \vee q$

(S93) $\neg p \mathcal{W} q \vee q \mathcal{W} \neg p \;\equiv\; p \Rightarrow q$

(S94) $(\neg p \mathcal{W} q \vee q \mathcal{W} \neg p) \wedge (\neg q \mathcal{W} p \vee p \mathcal{W} \neg q) \;\equiv\; (p \;\equiv\; q)$

(S95) $q \mathcal{U} (p \wedge q) \wedge p \mathcal{U} (p \wedge q) \;\equiv\; p \wedge q$

(S96) $q \mathcal{W} (p \wedge q) \wedge p \mathcal{W} (p \wedge q) \wedge \Diamond (p \wedge q) \;\equiv\; p \wedge q$

(S97) $p \mathcal{W} (p \wedge q) \wedge q \mathcal{W} (p \wedge q) \;\equiv\; p \wedge q$

Proof and Variations of the Dummett Formula

(S98) **Dummett implicit:** $\Box ((p \Rightarrow \Box p) \Rightarrow \Box p) \Rightarrow (\Diamond \Box p \Rightarrow \Box p)$

(S99) **Dummett explicit:** $\Box ((p \Rightarrow \bigcirc p) \wedge (\Box (p \Rightarrow \Box p) \Rightarrow \Box p)) \Rightarrow (\Diamond \Box p \Rightarrow \Box p)$

(S100) **Dummett variant:** $\Box (\Diamond (p \Rightarrow \Box p) \Rightarrow \Box p) \Rightarrow (\Diamond \Box p \Rightarrow \Box p)$

(S101) **Dummett variant:** $\Box (\Box (p \Rightarrow \Box p) \Rightarrow \Box p) \Rightarrow (\Box (p \Rightarrow \bigcirc p) \Rightarrow \Box p)$