CS243: Discrete Structures

Functions

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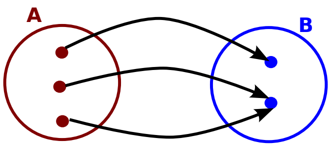
Functions

- A function f from a set A to a set B assigns each element of A to exactly one element of B.
- A is called domain of f, and B is called codomain of f.
- If f maps element $a \in A$ to element $b \in B$, we write $f(a) = b$
- If $f(a) = b$, b is called image of a; a is called preimage of b.
- Range of f is the set of all images of elements in A.

Functions Examples and Non-Examples

Is this mapping a function?



Functions Examples and Non-Examples

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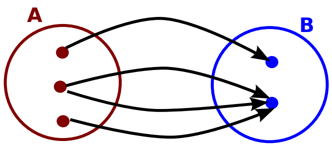

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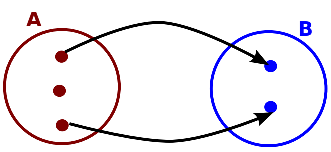

Functions Examples and Non-Examples

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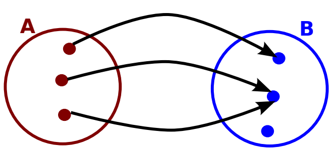
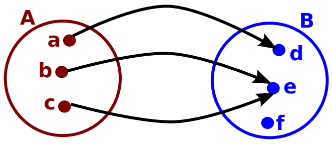


Function Terminology Examples



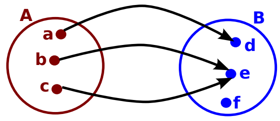
- What is the range of this function?
- What is the image of c?
- What is the preimage of e?

Image of a Set

- We can extend the definition of image to a set
- Suppose f is a function from A to B and S is a subset of A
- The image of S under f includes exactly those elements of B that are images of elements of S:

$$f(S) = \{t \mid \exists s \in S.\ t = f(s)\}$$

- What is the image of $\{b, c\}$?



One-to-One Functions

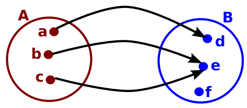
- A function f is called one-to-one if and only if $f(x) = f(y)$ implies $x = y$ for every x, y in the domain of f:

$$\forall x, y.\ (f(x) = f(y) \rightarrow x = y)$$

- One-to-one functions never assign different elements in the domain to the same element in the codomain:

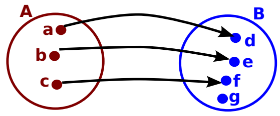
$$\forall x, y.\ (x \neq y \rightarrow f(x) \neq f(y))$$

- A one-to-one function also called injection or injective function
- Is this function one-to-one?



More Injective Function Examples

- Is this function injective?



- Consider the function $f(x) = x^2$ from set of integers to set of integers. Is this injective?
- What about if the domain of f is the set of non-negative integers?

Proving Injectivity Example

- Consider the function f from $\mathbb{Z}$ to $\mathbb{Z}$ defined as:

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \geq 0 \\ -3x + 2 & \text{if } x < 0 \end{cases}$$

- Prove that f is injective.
- We need to show that if $x \neq y$, then $f(x) \neq f(y)$
- What proof technique do we need to use?

Proving Injectivity Example, cont.

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \geq 0 \\ -3x + 2 & \text{if } x < 0 \end{cases}$$

- Case 1: $x \geq 0$, $y \geq 0$
- Let's use proof by contradiction.
- Suppose $x \neq y$, but $f(x) = f(y)$
- Since $x, y \geq 0$, $f(x) = 3x + 1$ and $f(y) = 3y + 1$
- Since we assume $f(x) = f(y)$, this implies $3x + 1 = 3y + 1$
- But this implies $x = y$, a contradiction.

Proving Injectivity Example, cont.

$$f(x) = \begin{cases} 3x+1 & \text{if } x \geq 0 \\ -3x+2 & \text{if } x < 0 \end{cases}$$

- Case 2: $x \geq 0$, $y < 0$
- For contradiction, suppose $x \neq y$, but $f(x) = f(y)$
- Since $x \geq 0$, $f(x) = 3x + 1$
- Since $y < 0$, $f(y) = -3y + 2$
- Since we assume $f(x) = f(y)$, this implies $3x + 1 = -3y + 2$
- But the equation $3x + 3y = 1$ has no solutions over integers, thus $f(x) \neq f(y)$

Proving Injectivity Example, cont.

$$f(x) = \begin{cases} 3x+1 & \text{if } x \geq 0 \\ -3x+2 & \text{if } x < 0 \end{cases}$$

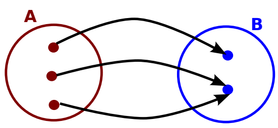
- Case 3: $y \geq 0$, $x < 0$
- This proof is same as proof of case 2.
- Case 4: $x < 0, y < 0$
- Suppose $x \neq y$ but $f(x) = f(y)$
- Then, $-3x + 2 = -3y + 2$, but this implies $x = y$, a contradiction.

Onto Functions

- A function f from A to B is called onto iff for every element $y \in B$, there is an element $x \in A$ such that $f(x) = y$:

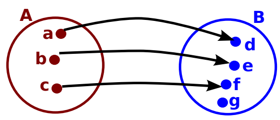
$$\forall y.\ (y \in B \rightarrow (\exists x.\ (x \in A \land f(x) = y))$$

- Onto functions also called surjective functions or surjections
- Another way of desrcibing onto functions: range and codomain are the same
- Is this function onto?



Examples of Onto Functions

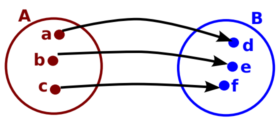
- Is this function onto?



- Consider the function $f(x) = x^2$ from the set of integers to the set of integers. Is f surjective?

Bijective Functions

- Function that is both onto and one-to-one called bijection
- Bijection also called one-to-one correspondence or invertible function
- Example of bijection:



Bijection Example

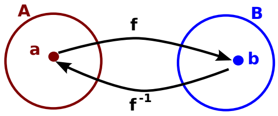
- The identity function I on a set A is the function that assigns every element of A to itself, i.e., $\forall x \in A.\ I(x) = x$
- Prove that the identity function is a bijection.
- Need to prove I is both one-to-one and onto.
- One-to-one: We need to show $\forall x, y.\ (x \neq y \rightarrow I(x) \neq I(y))$
- Suppose $x \neq y$.
- Since $I(x) = x$ and $I(y) = y$, and $x \neq y$, $I(x) \neq I(y)$

Bijection Example, cont.

- Now, prove I is onto, i.e., for every b, there exists some a such that $f(a) = b$
- For contradiction, suppose there is some b such that $\forall a \in A.\ I(a) \neq b$
- Since $I(a) = a$, this means $\forall a \in A.\ a \neq b$
- But since b is itself in A, this would imply $b \neq b$, yielding a contradiction.
- Since I is both onto and one-to-one, it is a bijection. $\square$

Inverse Functions

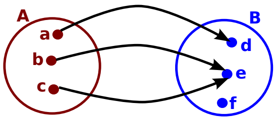
- Every bijection from set A to set B also has an inverse function
- The inverse of bijection f, written f^{-1}, is the function that assigns to $b \in B$ a unique element $a \in A$ such that $f(a) = b$



- Observe: Inverse functions are only defined for bijections, not arbitrary functions!
- This is why bijections are also called invertible functions

Why are Inverse Functions Only Defined on Bijections?

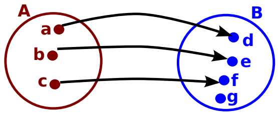
- Suppose f is not injective, i.e., assigns distinct elements to the same element.



- Then, the inverse is not a function because it assigns the same element to distinct elements

Why are Inverse Functions Only Defined on Bijections?

- Suppose f is not surjective, i.e., range and codomain are not the same



- Then, the inverse is not a function because it does not assign some element in B to any element in A
- Hence, inverse functions only defined for bijections!

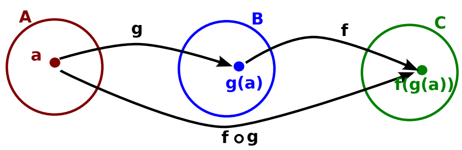
Inverse Function Examples

- Let f be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = x^2$. Is f invertible?
- Let g be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $g(x) = x + 1$. Is g invertible?
- What is g^{-1}?

Function Composition

- Let g be a function from A to B, and f from B to C.
- The composition of f and g, written $f \circ g$, is defined by:

$$(f \circ g)(x) = f(g(x))$$



Composition Example

- Let f and g be function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = 2x + 3$ and $g(x) = 3x + 2$
- What is $f \circ g$?

Another Composition Example

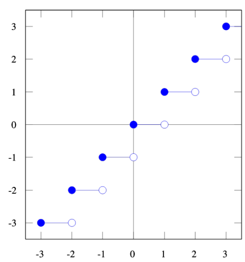
- Prove that $f^{-1} \circ f = I$ where I is the identity function.
- Since $I(x) = x$, need to show $(f^{-1} \circ f)(x) = x$
- First, $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
- Let $f(x)$ be y
- Then, $f^{-1}(f(x)) = f^{-1}(y)$
- By definition of inverse, $f^{-1}(y) = x$ iff $f(x) = y$
- Thus, $f^{-1}(f(x)) = f^{-1}(y) = x$ □

Example

- Prove that if f and g are injective, then $f \circ g$ is also injective.

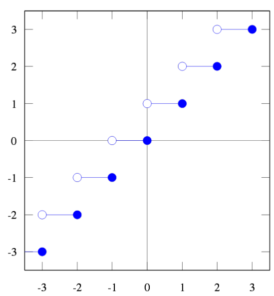
Floor and Ceiling Functions

- Two important functions in discrete math are floor and ceiling functions, both from $\mathbb{R}$ to $\mathbb{Z}$
- The floor of a real number x, written $\lfloor x \rfloor$, is the largest integer less than or equal to x.

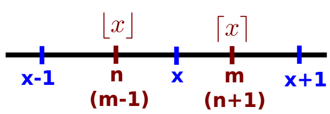


Ceiling Function

- The ceiling of a real number x, written $\lceil x \rceil$, is the smallest integer greater than or equal to x.

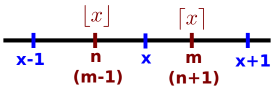


Useful Properties of Floor and Ceiling Functions



1. For integer n and real number x, $\lfloor x \rfloor = n$ iff $n \leq x < n + 1$
2. For integer n and real number x, $\lceil x \rceil = m$ iff $m - 1 < x \leq m$
3. For any real x, $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

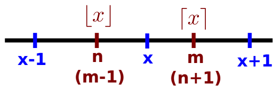
Proofs about Floor/Ceiling Functions



Prove that $\lfloor -x \rfloor = -\lceil x \rceil$

- Let m be $\lceil x \rceil$. Then, by property (2), $m-1 < x \leq m$
- Multiple both sides by -1: $-m+1 > -x \geq -m$
- Rewrite this as: $-m \leq -x < -m+1$
- Now, by property (1), we have $\lfloor -x \rfloor = -m$
- Thus, $\lfloor -x \rfloor = -m = -\lceil x \rceil$ □

Another Example



Prove that $\lfloor x+k \rfloor = \lfloor x \rfloor + k$ where k is an integer

- Let n be $\lfloor x \rfloor$. Then, by property (1), $n \leq x < n+1$
- Add k to both sides: $n+k \leq x+k < n+k+1$
- Again, by property (1), $\lfloor x+k \rfloor = n+k$
- Since $\lfloor x \rfloor = n$, $\lfloor x \rfloor + k$ is also $n+k$
- Hence, $\lfloor x+k \rfloor = \lfloor x \rfloor + k$ □

More Examples

Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

- Observe: Any real number x can be written as $n + \epsilon$ where $n = \lfloor x \rfloor$ and $0 \leq \epsilon < 1$
- To prove desired property, do proof by cases
- Case 1: $0 \leq \epsilon < \frac{1}{2}$
- Case 2: $\frac{1}{2} \leq \epsilon < 1$
- First prove property for first case, then second case

Proof of Case 1

Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

- Let x be $n + \epsilon$ where $0 \leq \epsilon < \frac{1}{2}$
- Then, $2x = 2n + 2\epsilon$ where $0 \leq 2\epsilon < 1$
- Hence, $\lfloor 2x \rfloor = 2n$
- Furthermore, $x + \frac{1}{2} = n + \epsilon + \frac{1}{2}$
- Since $\epsilon + \frac{1}{2} < 1$, $\lfloor x + \frac{1}{2} \rfloor = n$
- Thus, $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ is also $2n = \lfloor 2x \rfloor$

Proof of Case 2

Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

- Let x be $n + \epsilon$ where $\frac{1}{2} \leq \epsilon < 1$
- Then, $2x = 2n + 2\epsilon$ where $1 \leq 2\epsilon < 2$
- Hence, $2x = 2n + 1 + \epsilon'$ where $0 \leq \epsilon' < 1$
- Thus, $\lfloor 2x \rfloor = 2n + 1$
- Furthermore, $x + \frac{1}{2} = n + \epsilon + \frac{1}{2} = n + 1 + \epsilon''$ $(0 \leq \epsilon'' < 1)$
- Thus, $\lfloor x + \frac{1}{2} \rfloor = n + 1$
- Thus, $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ is also $n + n + 1 = 2n + 1 = \lfloor 2x \rfloor$ □