

CS311H: Discrete Mathematics

First Order Logic, Rules of Inference

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Review of Last Lecture

- ▶ What are the building blocks in FOL?
- ▶ What is an interpretation for a FOL formula?
- ▶ What is DeMorgan's law for quantifiers?

Translating First-Order Logic into English

Given predicates $student(x)$, $atUT(x)$, and $friends(x, y)$, what do the following formulas say in English?

- ▶ $\forall x. ((atUT(x) \wedge student(x)) \rightarrow (\exists y. (friends(x, y) \wedge \neg atUT(y))))$
- ▶ $\forall x. ((student(x) \wedge \neg atUT(x)) \rightarrow \neg \exists y. friends(x, y))$
- ▶ $\forall x. \forall y. ((student(x) \wedge student(y) \wedge friends(x, y)) \rightarrow (atUT(x) \wedge atUT(y)))$

Translating English into First-Order Logic

Given predicates $student(x)$, $atUT(x)$, and $friends(x, y)$, how do we express the following in first-order logic?

- ▶ "Every UT student has a friend"
- ▶ "At least one UT student has no friends"
- ▶ "All UT students are friends with each other"

Satisfiability, Validity in FOL

- ▶ An FOL formula F is **satisfiable** if there exists some interpretation such that F evaluates to true
- ▶ **Example:** Prove that $\forall x. (P(x) \rightarrow Q(x))$ is satisfiable.
- ▶ An FOL formula F is **valid** if, for all interpretations, F evaluates to true
- ▶ Prove that $\forall x. (P(x) \rightarrow Q(x))$ is not valid.
- ▶ Formulas that are satisfiable, but not valid are **contingent**, e.g., $\forall x. (P(x) \rightarrow Q(x))$

Example

- ▶ Is the following formula valid, unsat, or contingent? Prove your answer.

$$((\exists x. P(x)) \wedge (\exists x. Q(x))) \rightarrow (\exists x. (P(x) \wedge Q(x)))$$

Equivalence

- ▶ Two formulas F_1 and F_2 are equivalent if $F_1 \leftrightarrow F_2$ is valid
- ▶ In PL, we could prove equivalence using truth tables, but not possible in FOL
- ▶ However, we can still use known equivalences to rewrite one formula as the other
- ▶ Example: Prove that $\neg(\forall x. (P(x) \rightarrow Q(x)))$ and $\exists x. (P(x) \wedge \neg Q(x))$ are equivalent.
- ▶ Example: Prove that $\neg\exists x.\forall y.P(x, y)$ and $\forall x.\exists y.\neg P(x, y)$ are equivalent.

Rules of Inference

- ▶ We can prove validity in FOL by using **proof rules**
- ▶ Proof rules are written as **rules of inference**:

$$\frac{\begin{array}{l} \text{Hypothesis1} \\ \text{Hypothesis2} \\ \dots \end{array}}{\text{Conclusion}}$$

- ▶ An example inference rule:

$$\frac{\begin{array}{l} \text{All men are mortal} \\ \text{Socrates is a man} \end{array}}{\therefore \text{Socrates is mortal}}$$

- ▶ We'll learn about more general inference rules that will allow constructing **formal** proofs

Modus Ponens

- ▶ Most basic inference rule is **modus ponens**:

$$\frac{\begin{array}{l} \phi_1 \\ \phi_1 \rightarrow \phi_2 \end{array}}{\phi_2}$$

- ▶ Modus ponens applicable to both propositional logic and first-order logic

Example Uses of Modus Ponens

- ▶ Application of modus ponens in propositional logic:

$$\frac{\begin{array}{l} p \wedge q \\ (p \wedge q) \rightarrow r \end{array}}{r}$$

- ▶ Application of modus ponens in first-order logic:

$$\frac{\begin{array}{l} P(a) \\ P(a) \rightarrow Q(b) \end{array}}{Q(b)}$$

Modus Tollens

- ▶ Second important inference rule is **modus tollens**:

$$\frac{\begin{array}{l} \phi_1 \rightarrow \phi_2 \\ \neg\phi_2 \end{array}}{\neg\phi_1}$$

Example Uses of Modus Tollens

- ▶ Application of modus tollens in propositional logic:

$$\frac{\begin{array}{l} p \rightarrow (q \vee r) \\ \neg(q \vee r) \end{array}}{\neg p}$$

- ▶ Application of modus tollens in first-order logic:

$$\frac{\begin{array}{l} Q(a) \\ \neg P(a) \rightarrow \neg Q(a) \end{array}}{P(a)}$$

Hypothetical Syllogism (HS)

$$\frac{\phi_1 \rightarrow \phi_2}{\phi_2 \rightarrow \phi_3}$$

$$\frac{\phi_2 \rightarrow \phi_3}{\phi_1 \rightarrow \phi_3}$$

- ▶ Basically says "implication is transitive"
- ▶ Example:

$$\frac{P(a) \rightarrow Q(b)}{Q(b) \rightarrow R(c)}$$

Or Introduction and Elimination

- ▶ Or introduction:

$$\frac{\phi_1}{\phi_1 \vee \phi_2}$$

- ▶ Example application: "Socrates is a man. Therefore, either Socrates is a man or there are red elephants on the moon."

- ▶ Or elimination:

$$\frac{\phi_1 \vee \phi_2}{\neg \phi_2}$$

$$\phi_1$$

- ▶ Example application: "It is either a dog or a cat. It is not a dog. Therefore, it must be a cat."

And Introduction and Elimination

- ▶ And introduction:

$$\frac{\phi_1}{\phi_2}$$

$$\frac{\phi_1 \phi_2}{\phi_1 \wedge \phi_2}$$

- ▶ And elimination:

$$\frac{\phi_1 \wedge \phi_2}{\phi_1}$$

Resolution

- ▶ Final inference rule: resolution

$$\frac{\phi_1 \vee \phi_2}{\neg \phi_1 \vee \phi_3}$$

$$\phi_2 \vee \phi_3$$

- ▶ To see why this is correct, observe ϕ_1 is either true or false.
- ▶ Suppose ϕ_1 is true. Then, $\neg \phi_1$ is false. Therefore, by second hypothesis, ϕ_3 must be true.
- ▶ Suppose ϕ_1 is false. Then, by 1st hypothesis, ϕ_2 must be true.
- ▶ In any case, either ϕ_2 or ϕ_3 must be true; $\therefore \phi_2 \vee \phi_3$

Resolution Example

- ▶ Example 1:

$$\frac{P(a) \vee \neg Q(b)}{Q(b) \vee R(c)}$$

- ▶ Example 2:

$$\frac{p \vee q}{q \vee \neg p}$$

Summary

Name	Rule of Inference
Modus ponens	$\frac{\phi_1}{\phi_1 \rightarrow \phi_2}$
Modus tollens	$\frac{\phi_2}{\phi_1 \rightarrow \phi_2}$ $\frac{\neg \phi_2}{\neg \phi_1}$
Hypothetical syllogism	$\frac{\phi_1 \rightarrow \phi_2}{\phi_2 \rightarrow \phi_3}$ $\phi_1 \rightarrow \phi_3$
Or introduction	$\frac{\phi_1}{\phi_1 \vee \phi_2}$
Or elimination	$\frac{\phi_1 \vee \phi_2}{\neg \phi_2}$ ϕ_1
And introduction	$\frac{\phi_1}{\phi_2}$ $\phi_1 \wedge \phi_2$
And elimination	$\frac{\phi_1 \wedge \phi_2}{\phi_1}$
Resolution	$\frac{\phi_1 \vee \phi_2}{\neg \phi_1 \vee \phi_3}$ $\phi_2 \vee \phi_3$

Using the Rules of Inference

Assume the following hypotheses:

1. It is not sunny today and it is colder than yesterday.
2. We will go to the lake only if it is sunny.
3. If we do not go to the lake, then we will go hiking.
4. If we go hiking, then we will be back by sunset.

Show these lead to the conclusion: "We will be back by sunset."

Encoding in Logic

- ▶ First, encode hypotheses and conclusion as logical formulas.
- ▶ To do this, identify propositions used in the argument:
 - ▶ s = "It is sunny today"
 - ▶ c = "It is colder than yesterday"
 - ▶ l = "We'll go to the lake"
 - ▶ h = "We'll go hiking"
 - ▶ b = "We'll be back by sunset"

Encoding in Logic, cont.

- ▶ "It's not sunny today and colder than yesterday."
- ▶ "We will go to the lake only if it is sunny"
- ▶ "If we do not go to the lake, then we will go hiking."
- ▶ "If we go hiking, then we will be back by sunset."
- ▶ **Conclusion:** "We'll be back by sunset"

Formal Proof Using Inference Rules

1. $\neg s \wedge c$ Hypothesis
2. $l \rightarrow s$ Hypothesis
3. $\neg l \rightarrow h$ Hypothesis
4. $h \rightarrow b$ Hypothesis

Another Example

Assume the following hypotheses:

1. It is not raining or Kate has her umbrella
2. Kate does not have her umbrella or she does not get wet
3. It is raining or Kate does not get wet
4. Kate is grumpy only if she is wet

Show these lead to the conclusion: "Kate is not grumpy."

Encoding in Logic

- ▶ First, encode hypotheses and conclusion as logical formulas.
- ▶ To do this, identify propositions used in the argument:
 - ▶ r = "It is raining"
 - ▶ u = "Kate has her umbrella"
 - ▶ w = "Kate is wet"
 - ▶ g = "Kate is grumpy"

Encoding in Logic, cont.

- ▶ "It is not raining or Kate has her umbrella."
- ▶ "Kate does not have her umbrella or she does not get wet"
- ▶ "It is raining or Kate does not get wet."
- ▶ " Kate is grumpy only if she is wet."
- ▶ **Conclusion:** "Kate is not grumpy."

Formal Proof Using Inference Rules

1. $\neg r \vee u$ Hypothesis
2. $\neg u \vee \neg w$ Hypothesis
3. $r \vee \neg w$ Hypothesis
4. $g \rightarrow w$ Hypothesis

Additional Inference Rules for Quantified Formulas

- ▶ Inference rules we learned so far are sufficient for reasoning about quantifier-free statements
- ▶ Four more inference rules for making deductions from quantified formulas
- ▶ These come in pairs for each quantifier (universal/existential)
- ▶ One is called **generalization**, the other one called **instantiation**

Universal Instantiation

- ▶ If we know something is true for all members of a group, we can conclude it is also true for a **specific** member of this group
- ▶ This idea is formally called **universal instantiation**:

$$\frac{\forall x.P(x)}{P(c)} \text{ (for any } c\text{)}$$

- ▶ If we know "All CS classes at UT are hard", universal instantiation allows us to conclude "CS311 is hard"!

Example

- ▶ Consider predicates $\text{man}(x)$ and $\text{mortal}(x)$ and the hypotheses:
 1. All men are mortal:
 2. Socrates is a man:
- ▶ Using rules of inference, prove $\text{mortal}(\text{Socrates})$

Universal Generalization

- ▶ Suppose we can prove a claim for an **arbitrary** element in the domain.
- ▶ Since we've made no assumptions about this element, proof should apply to all elements in the domain.
- ▶ This correct reasoning is captured by **universal generalization**

$$\frac{P(c) \text{ for arbitrary } c}{\forall x.P(x)}$$

Example

Prove $\forall x.Q(x)$ from the hypotheses:

1. $\forall x.(P(x) \rightarrow Q(x))$ Hypothesis
2. $\forall x.P(x)$ Hypothesis
3. $P(c) \rightarrow Q(c)$ \forall -inst (1)
4. $P(c)$ \forall -inst (2)
5. $Q(c)$ Modus ponens (3), (4)
6. $\forall x.Q(x)$ \forall -gen (5)

Caveat About Universal Generalization

- ▶ When using universal generalization, need to ensure that c is truly arbitrary!
- ▶ If you prove something about a specific person **Mary**, you cannot make generalizations about all people

$$\frac{\text{even}(2)}{\forall x.\text{even}(x)}$$

Existential Instantiation

- ▶ Consider formula $\exists x.P(x)$.
- ▶ We know there is some element, say c , in the domain for which $P(c)$ is true.

- ▶ This is called **existential instantiation**:

$$\frac{\exists x.P(x)}{P(c)} \text{ (for unused } c\text{)}$$

- ▶ Here, c is a **fresh** name (i.e., not used before in proof).
 - ▶ Otherwise, can prove non-sensical things such as: "There exists some animal that can fly. Thus, rabbits can fly!"

Example Using Existential Instantiation

Consider the hypotheses $\exists x.P(x)$ and $\forall x.\neg P(x)$. Prove that we can derive a contradiction (i.e., **false**) from these hypotheses.

1. $\exists x.P(x)$ Hypothesis
2. $\forall x.\neg P(x)$ Hypothesis
- 3.
- 4.
- 5.
- 6.

Existential Generalization

- ▶ Suppose we know $P(c)$ is true for some constant c
- ▶ Then, there exists an element for which P is true
- ▶ Thus, we can conclude $\exists x.P(x)$
- ▶ This inference rule called **existential generalization**:

$$\frac{P(c)}{\exists x.P(x)}$$

Example Using Existential Generalization

Consider the hypotheses $atUT(\text{George})$ and $smart(\text{George})$. Prove $\exists x.(atUT(x) \wedge smart(x))$

1. $atUT(\text{George})$ Hypothesis
2. $smart(\text{George})$ Hypothesis
- 3.
- 4.

Summary of Inference Rules for Quantifiers

Name	Rule of Inference
Universal Instantiation	$\frac{\forall x.P(x)}{P(c)} \text{ (any } c\text{)}$
Universal Generalization	$\frac{P(c) \text{ (for arbitrary } c\text{)}}{\forall x.P(x)}$
Existential Instantiation	$\frac{\exists x.P(x)}{P(c) \text{ for fresh } c}$
Existential Generalization	$\frac{P(c)}{\exists x.P(x)}$

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Example I

► Prove that these hypotheses imply $\exists x.(P(x) \wedge \neg B(x))$:

1. $\exists x.(C(x) \wedge \neg B(x))$ (Hypothesis)
2. $\forall x.(C(x) \rightarrow P(x))$ (Hypothesis)

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Example II

► Prove the below hypotheses are contradictory by deriving **false**

1. $\forall x.(P(x) \rightarrow (Q(x) \wedge S(x)))$ (Hypothesis)
2. $\forall x.(P(x) \wedge R(x))$ (Hypothesis)
3. $\exists x.(\neg R(x) \vee \neg S(x))$ (Hypothesis)

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Example III

Prove $\exists x.father(x, Evan)$ from the following premises:

1. $\forall x.\forall y.((parent(x, y) \wedge male(x)) \rightarrow father(x, y))$
2. $parent(Tom, Evan)$
3. $male(Tom)$

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