Maximum Satisfiability

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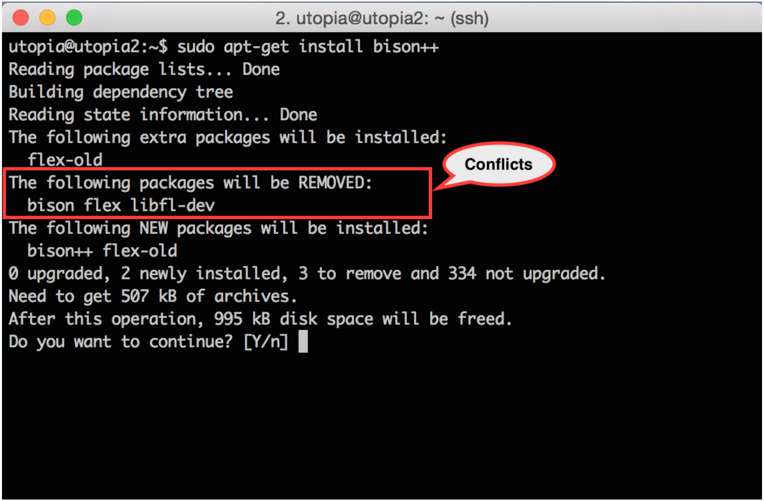
What is Boolean Satisfiability?

- **Fundamental problem** in Computer Science
  - The first problem to be proven NP-Complete
  - Has a wide range of applications
- Formula:
  - $\varphi = (\neg x_2 \vee \neg x_1) \wedge (x_2 \vee \neg x_3) \wedge (x_1) \wedge (x_3)$
- Boolean Satisfiability (SAT):
  - Is there an assignment of true or false values to variables such that φ evaluates to true?

Software Package Upgradeability Problem



Software Package Upgradeability Problem

Package	Dependencies	Conflicts
p_1	$\{p_2 \vee p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
p_4	$\{p_2 \wedge p_3\}$	$\{\}$

- Set of packages we want to install: $\{p_1, p_2, p_3, p_4\}$
- Each package p_i has a set of **dependencies**:
  - Packages that must be installed for p_i to be installed
- Each package p_i has a set of **conflicts**:
  - Packages that cannot be installed for p_i to be installed

NP Completeness

"I can't find an efficient algorithm, but neither can all these famous people."

- Giving up?
  - The problem is NP-hard, so let's develop heuristics or approximation algorithms.
- No! Current tools can find solutions for **very large** problems!

Software Package Upgradeability Problem as SAT

Package	Dependencies	Conflicts
p_1	$\{p_2 \lor p_3\}$	$\{p_4\}$
p_2	$\{p_3\}$	$\{\}$
p_3	$\{p_2\}$	$\{p_4\}$
p_4	$\{p_2 \land p_3\}$	$\{\}$

How can we encode this problem to Boolean Satisfiability?

Software Package Upgradeability Problem as SAT

Formula φ:

Dependencies	$\neg p_1 \vee p_2 \vee p_3$	$\neg p_2 \vee p_3$	$\neg p_3 \vee p_2$	
Conflicts	$\neg p_4 \vee p_2$	$\neg p_4 \vee p_3$	$\neg p_1 \vee \neg p_4$	$\neg p_3 \vee \neg p_4$
Packages	p_1	p_2	p_3	p_4

- Formula is unsatisfiable 
- We cannot install all packages
- How many packages can we install?

What is Maximum Satisfiability?

- Maximum Satisfiability (MaxSAT):
  - Clauses in the formula are either **soft** or **hard**
  - Hard clauses: **must** be satisfied
    (e.g. conflicts, dependencies)
  - Soft clauses: **desirable** to be satisfied
    (e.g. package installation)
- **Goal**: Maximize number of satisfied soft clauses

How to encode Software Package Upgradeability?

Software Package Upgradeability problem as MaxSAT:

- What are the hard constraints?
  - **(Hint)** Dependencies, conflicts or installation packages?
- What are the soft constraints?
  - **(Hint)** Dependencies, conflicts or installation packages?

How to encode Software Package Upgradeability?

Software Package Upgradeability problem as MaxSAT:

- What are the hard constraints?
  - Dependencies and conflicts
- What are the soft constraints?
  - Installation of packages

Software Package Upgradeability Problem as MaxSAT

MaxSAT Formula:

φ_h (Hard):	$\neg p_1 \vee p_2 \vee p_3$	$\neg p_2 \vee p_3$	$\neg p_3 \vee p_2$	
	$\neg p_4 \vee p_2$	$\neg p_4 \vee p_3$	$\neg p_1 \vee \neg p_4$	$\neg p_3 \vee \neg p_4$
φ_s (Soft):	p_1	p_2	p_3	p_4

- Dependencies and conflicts are encoded as hard clauses
- Installation of packages are encoded as soft clauses
- **Optimal solution** (3 out 4 packages are installed)

What is MaxSAT Complexity?

- Deciding whether k clauses can be satisfied: NP-Complete
  - Input: A CNF formula φ, a positive integer k
  - Question: Is there an assignment that satisfies at least k clauses in φ?
- MaxSAT is $\mathrm{FP}^{\mathrm{NP}}$-Complete
  - The class of binary relations $f(x, y)$ where given x we can compute y in polynomial time with access to an NP oracle
  - A SAT solver acts as the NP oracle most often in practice
- MaxSAT is hard to approximate (APX-Complete)
  - APX: class of NP optimization problems that:
    - admit a constant-factor approximation algorithm, *but*
    - have no poly-time approximation scheme (unless NP=P)

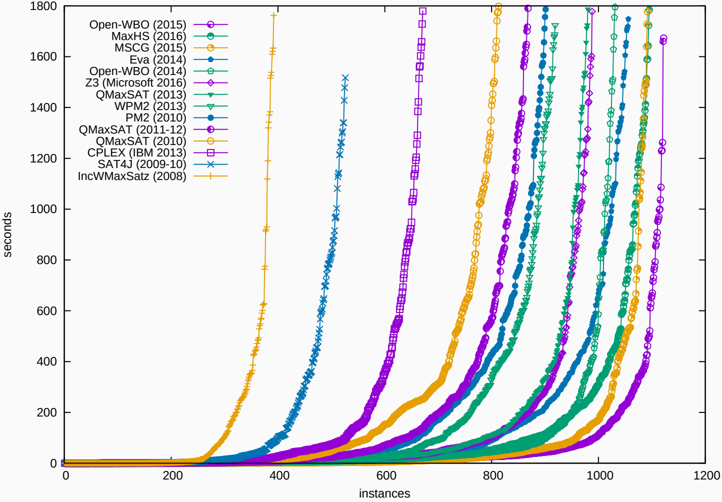
Why is MaxSAT Important?

- Many real-world applications can be encoded to MaxSAT:
  - Software package upgradeability
  - Error localization in C code

  - Haplotyping with pedigrees

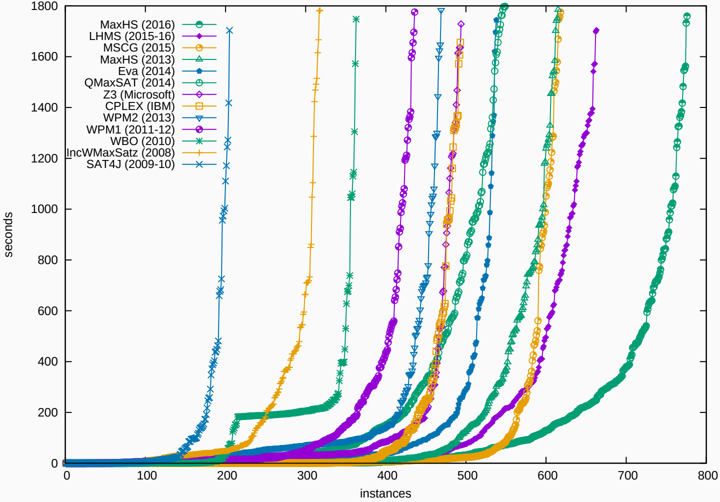
  - ...
- MaxSAT algorithms are **very effective** for solving real-word problems

The MaxSAT (r)evolution – Partial MaxSAT



- Best solver can solve 3× more benchmarks than in 2008!
- Better than tools like CPLEX (IBM) and Z3 (Microsoft)!

The MaxSAT (r)evolution – Partial Weighted MaxSAT

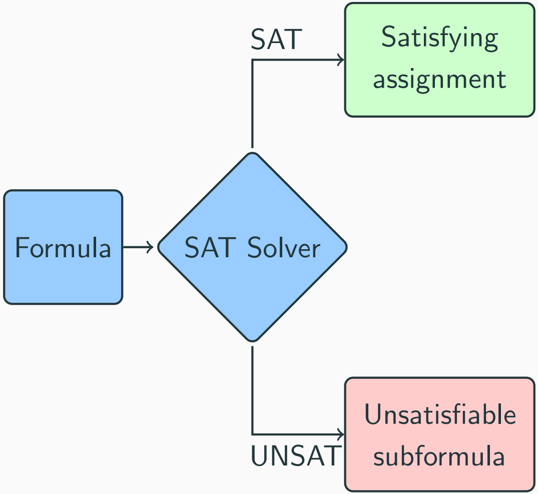


- Best solver can solve 2.5× more benchmarks than in 2008!
- Better than tools like CPLEX (IBM) and Z3 (Microsoft)!

Outline

- MaxSAT Algorithms:
  - **Upper bound search** on the number of unsatisfied soft clauses
  - **Lower bound search** on the number of unsatisfied soft clauses
- **Partitioning** in MaxSAT:
  - Use the structure of the problem to guide the search
- Using MaxSAT solvers

SAT Solvers


Formula
SAT Solver
SAT
Satisfying assignment
UNSAT
Unsatisfiable subformula

Satisfying assignment

Formula:

$$x_1 \qquad x_2 \vee \neg x_1 \qquad \neg x_3 \vee x_1 \qquad \neg x_3 \vee \neg x_1 \qquad x_2 \vee \neg x_3$$

$$x_1 \qquad x_2 \vee \neg x_1 \qquad \neg x_3 \vee x_1 \qquad \neg x_3 \vee \neg x_1 \qquad x_2 \vee \neg x_3$$

- Satisfying assignment:
  - Assignment to the variables that evaluates the formula to true
  - $\mu = \{x_1 = 1, x_2 = 1, x_3 = 0\}$

Unsatisfiable subformula

Formula:

$$x_1 \quad x_3 \quad x_2 \vee \neg x_1 \quad \neg x_3 \vee x_1 \quad \neg x_2 \vee \neg x_1 \quad x_2 \vee \neg x_3$$

$$x_1 \quad x_3 \quad x_2 \vee \neg x_1 \quad \neg x_3 \vee x_1 \quad \neg x_2 \vee \neg x_1 \quad x_2 \vee \neg x_3$$

- Formula is unsatisfiable
- Unsatisfiable subformula (core):
  - $\varphi' \subseteq \varphi$, such that φ' is unsatisfiable

MaxSAT Algorithms

- MaxSAT algorithms build on SAT solver technology
- MaxSAT algorithms use constraints not defined in causal form:
  - AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$
  - General cardinality constraints, $\sum_{j=1}^{n} x_j \leq k$
  - Pseudo-Boolean constraints, $\sum_{j=1}^{n} a_j x_j \leq k$
- Efficient encodings to CNF

CNF Encodings

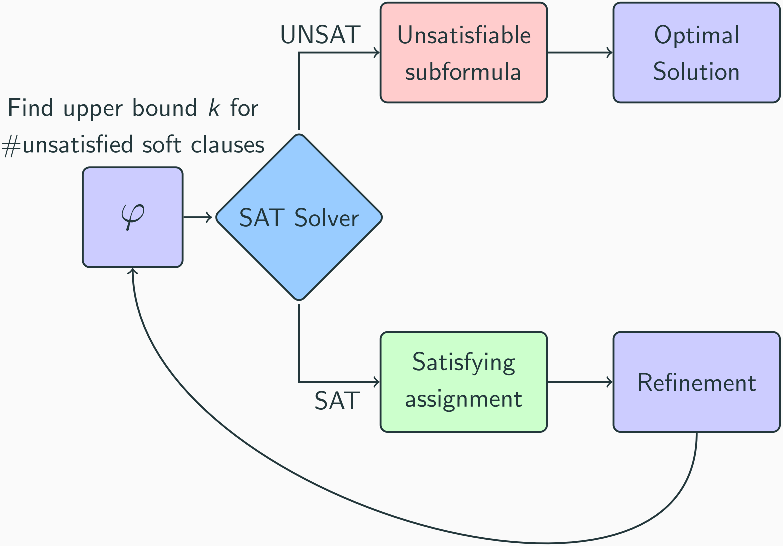
Sequential counters (Sinz [CP'05])

- AtMost1 constraints:
  - Clauses/Variables: $\mathcal{O}(n)$
- General cardinality constraints:
  - Clauses/Variables: $\mathcal{O}(n\ k)$

Sequential weighted counters (Hölldobler et al. [KI'12])

- Pseudo-Boolean constraints:
  - Clauses/Variables: $\mathcal{O}(n\ k)$

Upper Bound Search for MaxSAT



Can we unsatisfy less than 2 soft clauses? No!

Partial MaxSAT Formula:

φ_h:		$\neg x_2 \vee \neg x_1$	$x_2 \vee \neg x_3$		
φ_s:	x_1	x_3	$x_2 \vee \neg x_1$	$\neg x_3 \vee x_1$	

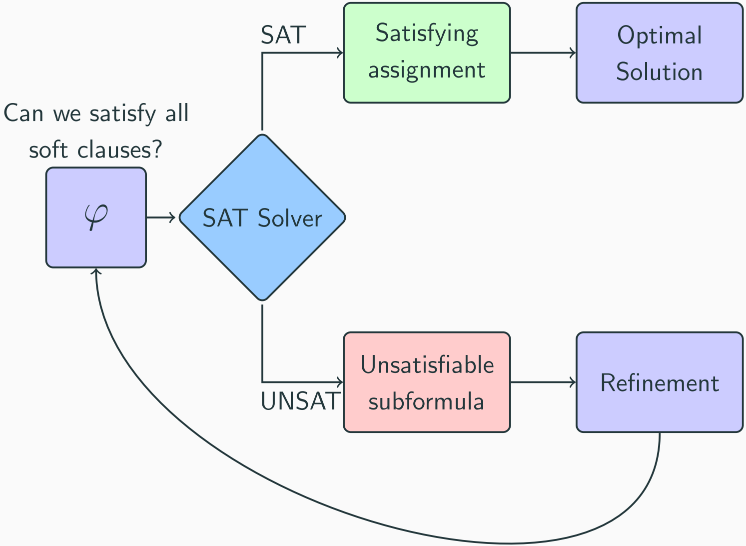
$\mu = 2 \qquad V_R = \{r_1, r_2, r_3, r_4\}$

- **Optimal solution**: given by the last model and corresponds to unsatisfying 2 soft clauses:
  - $\nu = \{x_1 = 1, x_2 = 0, x_3 = 0\}$

MaxSAT algorithms

- We have just seen a search on the **upper bound**
- What other kind of search can we do to find an optimal solution?
- What if we start searching from the **lower bound**?

Lower Bound Search for MaxSAT



Can we satisfy all soft clauses but 2? Yes!

Partial MaxSAT Formula:

$$\varphi_h: \quad \neg x_2 \vee \neg x_1 \qquad x_2 \vee \neg x_3 \qquad \mathrm{CNF}(\textstyle\sum_{r_i \in V_R} r_i \leq 2)$$

$$\varphi_s: \quad x_1 \vee r_1 \qquad x_3 \vee r_2 \qquad x_2 \vee \neg x_1 \vee r_3 \qquad \neg x_3 \vee x_1 \vee r_4$$

- Formula is satisfiable:
    - $\mu = \{x_1 = 1, x_2 = 0, x_3 = 0, r_1 = 0, r_2 = 1, r_3 = 1, r_4 = 0\}$
- Optimal solution unsatisfies 2 soft clauses

Unsatisfiability-based Algorithms

- What are the problems of this algorithm?
  (Hint) Number of relaxation variables? Size of the cardinality constraint? Other?

- We relax all soft clauses!
- The cardinality constraint contain as many literals as we have soft clauses!
- Can we do better?

Unsatisfiability-based Algorithms [MSU3: Marques-Silva&Planes'07]

Partial MaxSAT Formula:

$$\varphi_h: \quad \neg x_2 \vee \neg x_1 \qquad x_2 \vee \neg x_3 \qquad \mathrm{CNF}(r_1 + r_2 + r_3 + r_4 \leq 2)$$

$$\varphi_s: \quad x_1 \vee r_1 \qquad x_3 \vee r_2 \qquad x_2 \vee \neg x_1 \vee r_3 \qquad \neg x_3 \vee x$$

- Formula is satisfiable:
  - $\mu = \{x_1 = 1, x_2 = 0, x_3 = 0, r_1 = 0, r_2 = 1, r_3 = 1, r_4 = 0\}$
- Optimal solution unsatisfies 2 soft clauses

Unsatisfiability-based Algorithms

- What are the problems of this algorithm?
  (Hint) Number of relaxation variables? Size of the cardinality constraint? Other?

- We must translate cardinality constraints into CNF!
- If the number of literals is large than we may generate a **very large** formula!
- Can we do better?

Unsatisfiability-based Algorithms

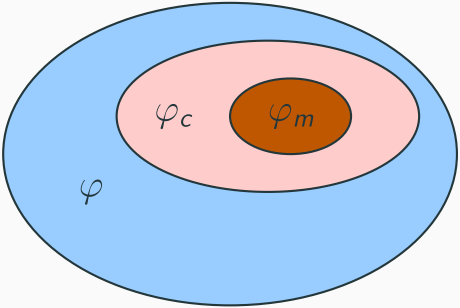
Partial MaxSAT Formula:

φ_h:		$\neg x_2 \vee \neg x_1$	$x_2 \vee \neg x_3$	
φ_s:	x_1	x_3	$x_2 \vee \neg x_1$	$\neg x_3 \vee x_1$

- Formula is satisfiable
- An optimal solution would be:
  - $\nu = \{x_1 = 1, x_2 = 0, x_3 = 0\}$
- This assignment unsatisfies 2 soft clauses

Challenges for Unsatisfiability-based MaxSAT Algorithms

- Unsatisfiable cores found by the SAT solver are **not minimal**



Formula φ

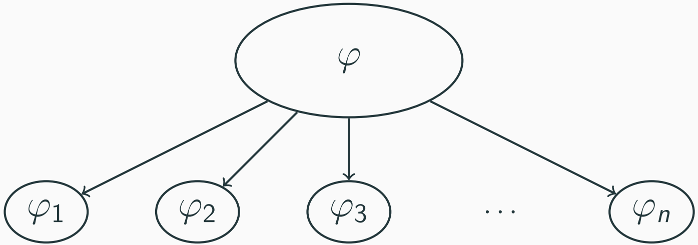
Unsatisfiable core φ_c

Minimal core φ_m

- **Minimizing** unsatisfiable cores is **computationally hard**

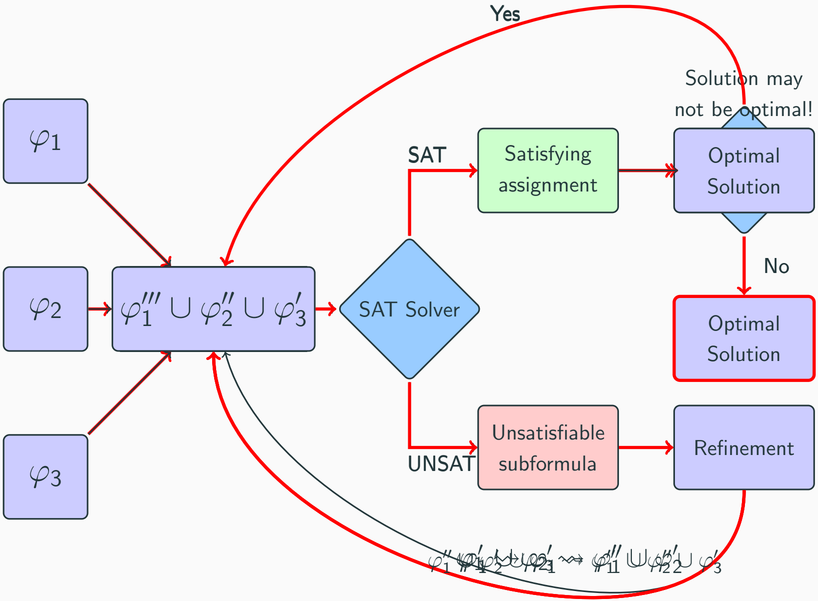
Partitioning in MaxSAT

- Partitioning in MaxSAT:
  - Partition the soft clauses into disjoint sets
  - Iteratively increase the size of the MaxSAT formula



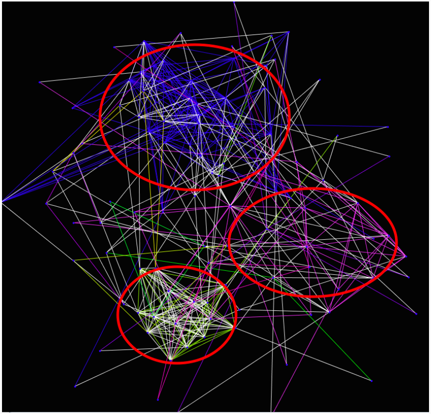
- Advantages:
  - **Easier formulas** for the SAT solver
  - **Smaller unsatisfiable cores** at each iteration

Framework for Partitioning-based MaxSAT Algorithms



How to Partition Soft Clauses?

- **Graph representation** of the MaxSAT formula:
  - Vertices: Variables
  - Edges: Between variables that appear in the same clause



Graph representations for MaxSAT

- There are many ways to represent MaxSAT as a graph:
  - Clause-Variable Incidence Graph (CVIG) [Martins et al. SAT 2013]
  - Variable Incidence Graph (VIG) [Martins et al. SAT 2013]
  - Hypergraph [Martins et al. SAT 2013]
  - **Resolution Graph** [Neves et al. SAT 2015]
  - ...

MaxSAT Formulas as Resolution-based Graphs

- MaxSAT solvers rely on the identification of **unsatisfiable cores**
- How can we capture sets of clauses that are closely related and are likely to result in unsatisfiable cores?
  - Represent MaxSAT formulas as **resolution graphs**!
  - Resolution graphs are based on the resolution rule
- Example of the resolution rule:

$$\frac{(x_1 \vee x_2) \quad (\neg x_2 \vee x_3)}{(x_1 \vee x_3)}$$

MaxSAT Formulas as Resolution-based Graphs

- Vertices: Represent each clause in the graph
- Edges: There is an edge between two vertices if you can apply the **resolution rule** between the corresponding clauses

Hard clauses:

$c_1 = x_1 \vee x_2$

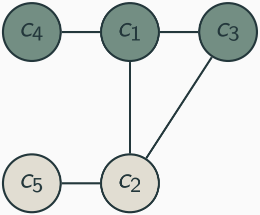
$c_2 = \neg x_2 \vee x_3$

$c_3 = \neg x_1 \vee \neg x_3$

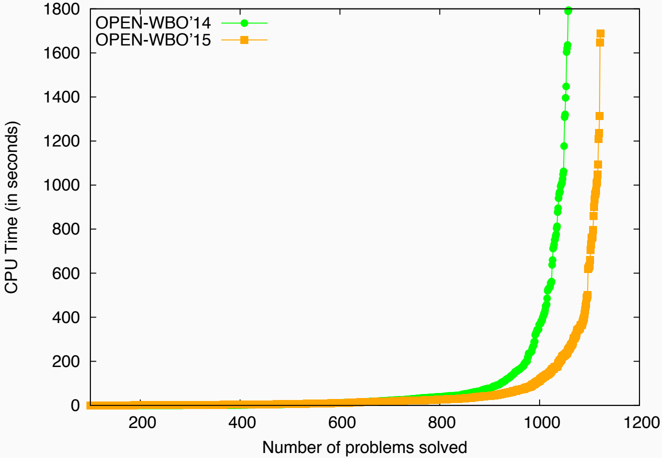
Soft clauses:

$c_4 = \neg x_1$

$c_5 = \neg x_3$



Impact of Partitioning in the MaxSAT Solving



- The techniques in Open-WBO have been **adopted** by other state-of-the-art MaxSAT solvers, e.g. MSCG, WPM

Want to try MaxSAT solving?

- Java:
  - **SAT4J**
  - http://www.sat4j.org/
- C++:
  - **Open-WBO**
  - Winner of multiples tracks in the MaxSAT Competition 2014, 2015 and 2016!
  - http://sat.inesc-id.pt/open-wbo/
- Annual competition:
  - http://www.maxsat.udl.cat/
  - Modify a solver today and enter this year competition!

Standard Solver Input Format: DIMACS WCNF

- Variables indexed from 1 to n
- Negation: -
  - -3 stands for $\neg x_3$
- 0: special end-of-line character
- One special header "p"-line:
  `p wcnf #vars #clauses top`
  - #vars: number of variables
  - #clauses: number of clauses
  - top: "weight" of hard clauses
- Clauses represented as lists of integers
  - Weight is the first number
  - $(\neg x_3 \vee x_1 \vee \neg x_4 5)$, weight 2:
    `2 -3 1 -45 0`
- Clause is hard if weight is equal to top

Standard Solver Input Format: DIMACS WCNF

Example: pointer analysis domain (pa-2.wcnf):

```
p wcnf 17997976 23364255 9223372036854775807
142 -11393180 12091478 0
200 -12496389 -1068725 13170751 0
209 -8854604 -8854942 -8854943 -8253894 9864153 0
174 -9406753 -8105076 11844088 0
200 -10403325 -8104972 12524177 0
142 -11987544 12096893 0
37 -10981341 -10980973 10838652 0
209 -9578314 -9579250 -9579251 -8254733 9578317 0
209 -8868994 -8870298 -8870299 -8254157 8868997 0
209 -9387012 -9387508 -9387509 -8253943 9387015 0
174 -9834074 -8106628 12074710 0
200 -10726788 -8105074 12909526 0
...
9223372036854775807 -13181184 0
9223372036854775807 -13181215 0
...  truncated 763 MB
```

Push-Button Solver Technology

Example: $ open-wbo pa-2.wcnf

```
c Open-WBO: a Modular MaxSAT Solver
c Version:  MaxSAT Evaluation 2016
c Authors:  Ruben Martins, Vasco Manquinho, Ines Lynce
c Contributors:  Miguel Neves, Saurabh Joshi, Mikolas Janota
...
c Problem Type:  Weighted
c Number of variables:  17,997,976
c Number of hard clauses:  8,237,870
c Number of soft clauses:  15,126,385
c Parse time:  5.60 s
...
o 4699
o 4609
o 143
s OPTIMUM FOUND
c Total time:  361.26 s v 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15...
...17997976
```

References

Cardinality and Pseudo-Boolean Encodings:

C. Sinz. Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. CP 2005: 827-831

N. Manthey, T. Philipp, P. Steinke. A More Compact Translation of Pseudo-Boolean Constraints into CNF Such That Generalized Arc Consistency Is Maintained. KI 2014: 123-134

T. Philipp, P. Steinke. PBLib - A Library for Encoding Pseudo-Boolean Constraints into CNF. SAT 2015: 9-16 `http://tools.computational-logic.org/content/pblib.php`

Community Structure:

C. Ansótegui, J. Giráldez-Cru, Jordi Levy. The Community Structure of SAT Formulas. SAT 2012: 410-423

Web pages of interest:

MaxSAT Evaluation: `http://www.maxsat.udl.cat/`
Open-WBO: `http://sat.inesc-id.pt/open-wbo/`
SAT4J: `http://www.sat4j.org/`
SATGraf: `https://bitbucket.org/znewsham/satgraf`