Efficient Interactive Proofs for Non-Deterministic Bounded Space

Joshua Cook (UT Austin) and
Ron Rothblum (Technion)

Background

Interactive Proofs (IPs)?

Untrusted Prover (Merlin)

Randomized Verifier (Arthur)

Many Questions

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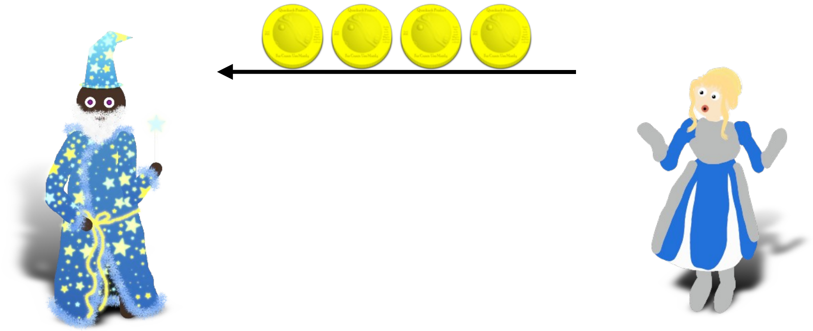

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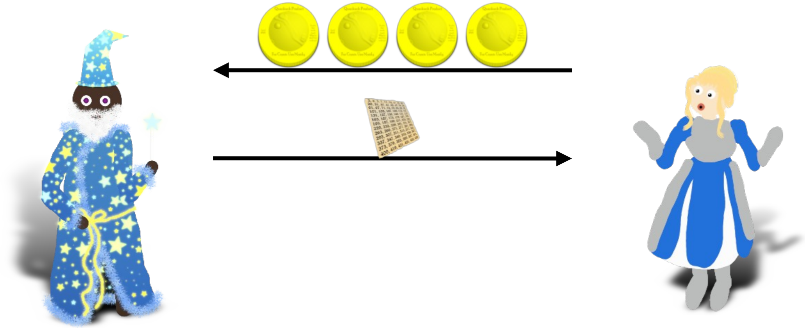

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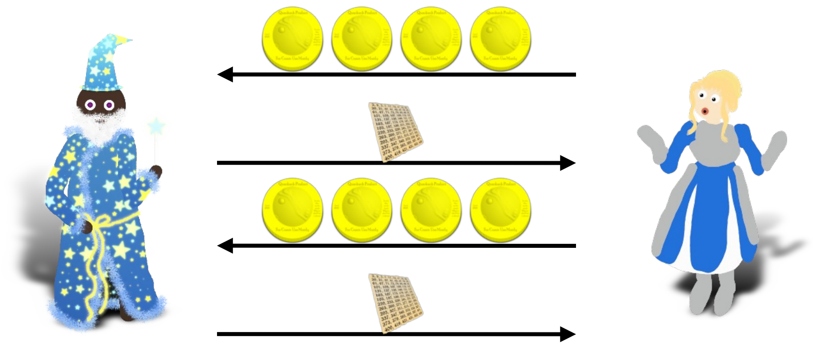

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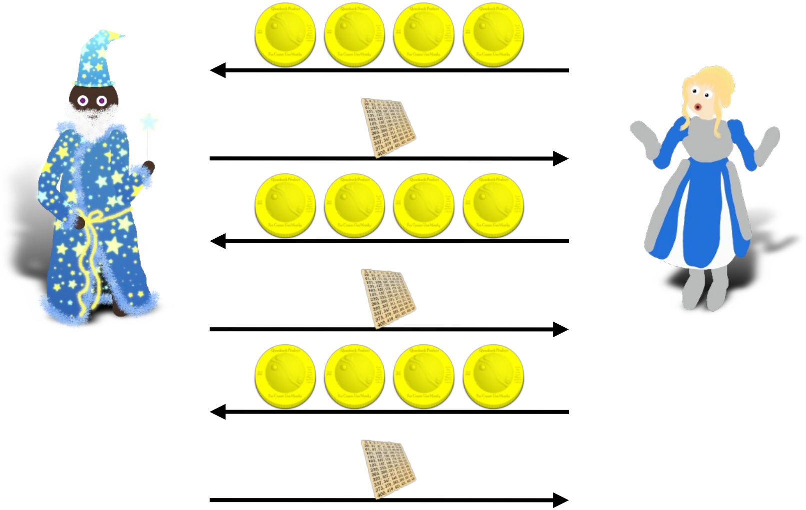

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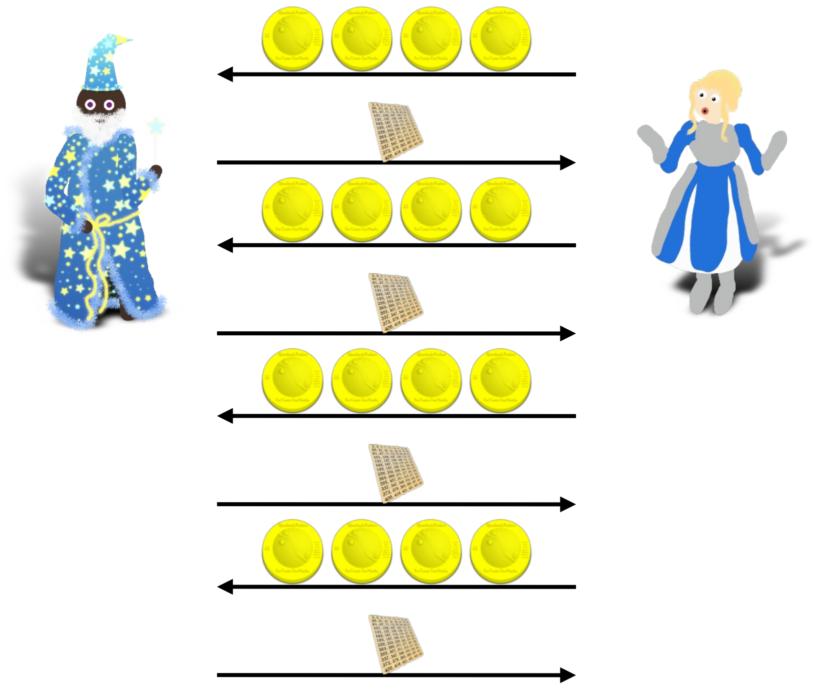

Goal

- Perfect Completeness: an honest Prover always succeeds
- Statistical Soundness: any Prover is unlikely to trick Verifier
- Primary goal: minimize Verifier time.
- Secondary goal: minimize Prover time, Verifier space, generalize to alternating algorithms

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  generalize to alternating algorithms

Other interactive protocols have much faster provers **[RRR16]**.

IPs for Deterministic Algorithms

TISP[T, S]: time T and space S algorithms.

ITIME[T]: IP with T Verifier time.

[LFKN90, Shamir90, GKR08, HMS13, Thaler20, Cook22]

$$\mathrm{TISP}[T, S] \subseteq \mathrm{ITIME}[\tilde{O}(n + S \log(T))]$$

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Natural generalization, extremely well studied.

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IP verifiers are programs with nondeterminism AND randomness.

- Using Nisan's [Nisan90] PRG, IPs speed up randomized algorithms as much as known for deterministic [Cook22].
- Should IPs speed up nondeterministic algorithms as much?

Randomness Time T, Space S

Nondeterminism
Time T, Space S

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Randomness Time T, Space S + Nondeterminism

Nondeterminism
Time T, Space S

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Randomness Time T, Space S + Nondeterminism → Randomness Nondeterminism Time $\tilde{O}(S\log(T))$, Space $\tilde{O}(S)$

Nondeterminism
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Nondeterminism Time T, Space S	+	Randomness	→	Randomness Nondeterminism Time ?, Space ?

Results

Result

Prior verifier for nondeterministic algorithms [Cook22]:

$$\mathrm{NTISP}[T, S] \subseteq \mathrm{ITIME}[\tilde{O}(n + S \log(T)^2)]$$

Our results:

$$\mathrm{NTISP}[T, S] \subseteq \mathrm{ITIME}[\tilde{O}(n + S \log(T))]$$

***Our verifiers also have space Õ(S)**

IPs for NTISP as fast as known for TISP, up to a log(S) factor.

Nondeterminism Time T, Space S	+	Randomness	→	Randomness Nondeterminism Time $S\log(T)^2$, Space $S\log(T)$

Result

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Nondeterminism
Time T, Space S + Randomness → Randomness Nondeterminism
Time $\tilde{O}(S\log(T))$, Space $\tilde{O}(S)$

Generalization

Generalizes to alternating algorithms with few alternations.

$$\mathrm{ATISP}_d[T, S] \subseteq \mathrm{ITIME}[\tilde{O}(n + S\log(T) + Sd)]$$

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$$\mathrm{ATISP}_d[T, S] \subseteq \mathrm{ITIME}[\tilde{O}(n + S\log(T) + Sd)]$$

- Closely related to depth d circuits **[Ruz81]**.
- Non-trivial to reduce to Alternating algorithms to nondeterministic ones.

Techniques for Prior Results

Computation Graph

View space S program as

2^S state graph, G

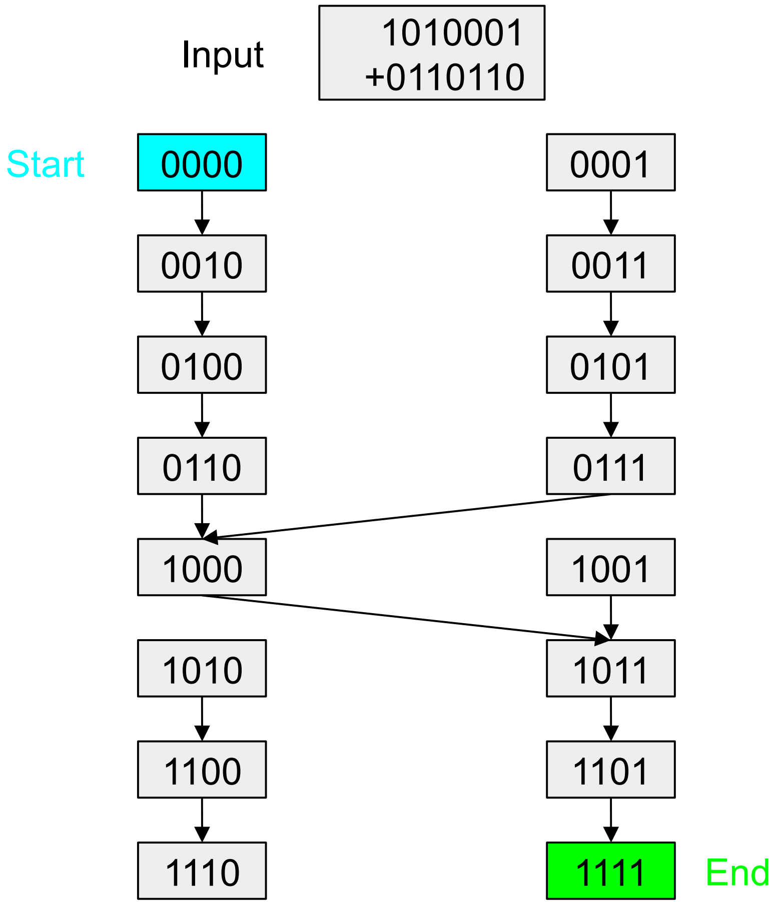
Edges are state transitions

Graph is a function of

Input, Program

Accepts IFF there is a length T path from start to end.

Edges are fast to compute



Adjacency Matrix For Deterministic Algorithms

$$M = \begin{bmatrix} & & 1 & & & & & \\ 1 & & & & & & & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

Represent G as an adjacency, M

Algorithm accepts in time T iff

$$M^T_{\text{start, end}} = 1$$

For deterministic algorithms, for all i, M^i is a binary matrix.

With sum check [Thaler14, HMS13]
can reduce $\mathbf{M}^2(x_0, x_1) = \alpha$

to $\mathbf{M}(y_0, y_1) = \beta$

In time $\tilde{O}(S)$.

Adjacency Matrix For Deterministic Algorithms

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$$M^4 = \begin{bmatrix} & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & & & 1 \\ & & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

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Nondeterministic Algorithm Matrix

$$M = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & 1 & 1 & & & & & \\ \hline
 & & 1 & 1 & & & & \\ \hline
 & & & 1 & 1 & & & \\ \hline
 & & & 1 & & & & \\ \hline
 & & & & & 1 & 1 & \\ \hline
 & & & & & & 1 & 1 \\ \hline
 & & & 1 & & & & 1 \\ \hline
 & & & & & & & 1 \\ \hline
\end{array}$$

For nondeterministic algorithms:

$M^T_{\text{start, end}}$ = #valid proofs

Matrix entries could be very large.

Random field size can help, but gives log(T) overhead of [Cook22].

Can’t use sums, need to use ORs.

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M =

	1	1					
		1	1				
			1	1			
			1				
					1	1	
						1	1
			1				1
							1

M^2 =

		1	2	1			
			2	1			
			1		1	1	
			1				
			1			1	2
			1				2
			1				1
							1

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						1	1
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							1

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		1	2	1			
			2	1			
			1		1	1	
			1				
			1			1	2
			1				2
			1				1
							1

M^4 =

			4		1	2	2
			3			1	2
			3				3
			1				
			2				3
			1				2
			1				1
							1

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		1	1				
			1	1			
			1				
					1	1	
						1	1
			1				1
							1

$M^2 =$

		1	2	1			
			2	1			
			1		1	1	
			1				
			1			1	2
			1				2
			1				1
							1

$M^4 =$

			4		1	2	2
			3			1	2
			3				3
			1				
			2				3
			1				2
			1				1
							1

$M^8 =$

			7				6
			4				3
			3				3
			1				
			2				3
			1				2
			1				1
							1

Boolean Formula for $M^{(2)}$

$$M = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & 1 & 1 & & & & & \\ \hline
 & & 1 & 1 & & & & \\ \hline
 & & & 1 & 1 & & & \\ \hline
 & & & 1 & & & & \\ \hline
 & & & & & 1 & 1 & \\ \hline
 & & & & & & 1 & 1 \\ \hline
 & & & 1 & & & & 1 \\ \hline
 & & & & & & & 1 \\ \hline
\end{array}$$

For notation, let $M(u,v) = M_{u,v}$

$$M^2(u,v) = \sum_{w\in\{0,1\}^S} M(u,w)\, M(w,v)$$

Replace + with $\vee$

$$M^{(2)}(u,v)=\bigvee_{w\in\{0,1\}^S} M(u,\, w)M(w,\, v)$$

Boolean Formula for $M^{(2)}$

$M =$

	1	1					
		1	1				
			1	1			
			1				
					1	1	
						1	1
			1				1
							1

$M^{(2)} =$

		1	1	1			
			1	1			
			1		1	1	
			1				
			1			1	1
			1				1
			1				1
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			1				
					1	1	
						1	1
			1				1
							1

$M^{(2)} =$

		1	1	1			
			1	1			
			1		1	1	
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			1			1	1
			1				1
			1				1
							1

$M^{(4)} =$

			1		1	1	1
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			1			1	1
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			1				1
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			1				1
							1

$M^{(8)} =$

			1				1
			1				1
			1				1
			1				
			1				1
			1				1
			1				1
							1

Reduction

Let **M** be the multilinear extension of M.

Given claim that $\mathbf{M}^{(2)}(u,v) = \alpha$, want to reduce to claim that $\mathbf{M}(u',v') = \beta$

Attempt 1, use $M^{(2)}(u,v) = 1 - \prod_{w\in\{0,1\}^S}(1 - M(u, w)\, M(w, v))$.

Degree is **way too high**: 2^S

Can handle one variable of w at a time with relinearizations [She92], but takes S linearizations of S variables, requires time $\tilde{O}(S^2)$.

Degree Reduction

$$1 = \bigvee \; 0\;0\;1\;0\;1\;1\;1\;0\;1\;0\;1\;0\;0$$

Razborov Smolensky

Razborov-Smolensky reduces degree

Idea: replace $\bigvee_{i \in [k]} x_i$

with $\sum_{j \in [k]} r_j \, x_j \bmod 2$

Where r is uniform random.

- Low degree, linear in GF(2).
- If all x are 0, outputs 0.
- If any x 1, then output 1 with probability 1/2.

$$1 = \bigvee \; 0 \; 0 \; 1 \; 0 \; 1 \; 1 \; 1 \; 0 \; 1 \; 0 \; 1 \; 0 \; 0$$

$$1 = \bigoplus \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \; 0$$

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```
1 = ∨ 0 0 1 0 1 1 1 0 1 0 1 0 0

1 = ⊕    0   0   1   0   0   0 0

0 = ⊕ 0   1 0 1 1     1 0     0
```

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1	=	$\vee$	0	0	1	0	1	1	1	0	1	0	1	0	0
1	=	$\oplus$		0		0		1		0		0		0	0
0	=	$\oplus$	0		1	0	1	1			1	0			0
0	=	$\oplus$		0		0		1	1	0	1		1	0	
1	=	$\oplus$							1	0	1	0	1	0	0
1	=	$\oplus$			1	0	1		1	0				0	0
0	=	$\oplus$		0	1		1		1	0		0	1		0

$$1 = \bigvee \; 0 \; 0 \; 1 \; 0 \; 1 \; 1 \; 1 \; 0 \; 1 \; 0 \; 1 \; 0 \; 0$$

Razborov Smolensky continued

Probability of failure decreases exponentially with repetitions.

$$\Pr_u[1-\prod_{i\in[L]}(1-\sum_{j\in[k]}r_{i,j}\, x_j)\neq\bigvee_{i\in[k]}x_i]\leq 2^{-L}$$

$m = 2^{O(S)}$ ANDs, choose $L = 2^{\ell} = \Omega(S)$. Most approximations are correct.

- Degree is small, $O(S)$
- $\ell = O(\log(S))$ variables

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$1 = \bigvee$ $\oplus$		0		0		1		0		0		0	0
$\oplus$	0		1	0	1	1			1	0			0
$\oplus$		0		0		1	1	0	1		1	0	
$\oplus$							1	0	1	0	1	0	0
$\oplus$		0	1		1		1	0		0	1		0
$\oplus$	0	0						0	1		1		

Applying Razborov Smolensky

Approximate $\bigvee_{w \in \{0,1\}^S} M(u, w)\, M(w, v)$

With $1 - \prod_{i \in [O(S)]} \left(1 - \sum_{w \in \{0,1\}^S} r_{i,w}\, M(u, w)\, M(w, v)\right)$

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Problem, too much randomness!!

Want seed length $O(S)$!

Completely random r has seed length $S\, 2^S$!!

Derandomization

Derandomization Step 1, Epsilon Biased Sets

Like [GR20], epsilon biased sets to sample r.

- Epsilon biased sets fool parity functions, Razborov-Smolensky IS a conjunction of parity functions!
- Epsilon biased sets equivalent to codes, easy to compute.

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$$1 - \prod_{i \in [O(S)]} (1 - \sum_{w \in \{0,1\}^S} D(seed_i, w)\, M(u, w)\, M(w, v))$$

Derandomization Step 1, Epsilon Biased Sets

Like [GR20], epsilon biased sets to sample r.

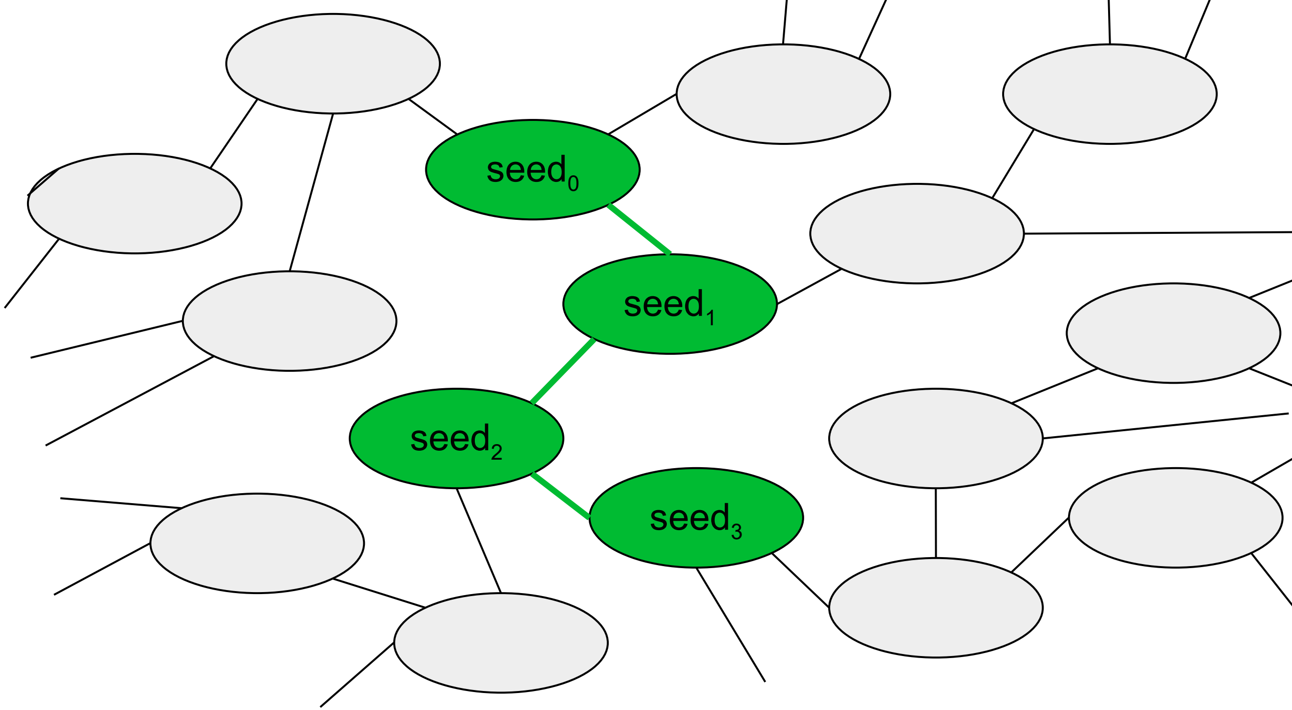
- Epsilon biased sets fool parity functions, Razborov-Smolensky IS a conjunction of parity functions!
- Epsilon biased sets equivalent to codes, easy to compute.

$$1 - \prod_{i \in [O(S)]} \left(1 - \sum_{w \in \{0,1\}^S} D(seed_i, w)\, M(u, w)\, M(w, v)\right)$$

- Takes O(S) bits for one seed for one choice of i: $seed_i$.
- But O(S) parities, $O(S^2)$ bits to sample them independently. Too much!

Derandomization Step 2, Random Walk Set Sampling

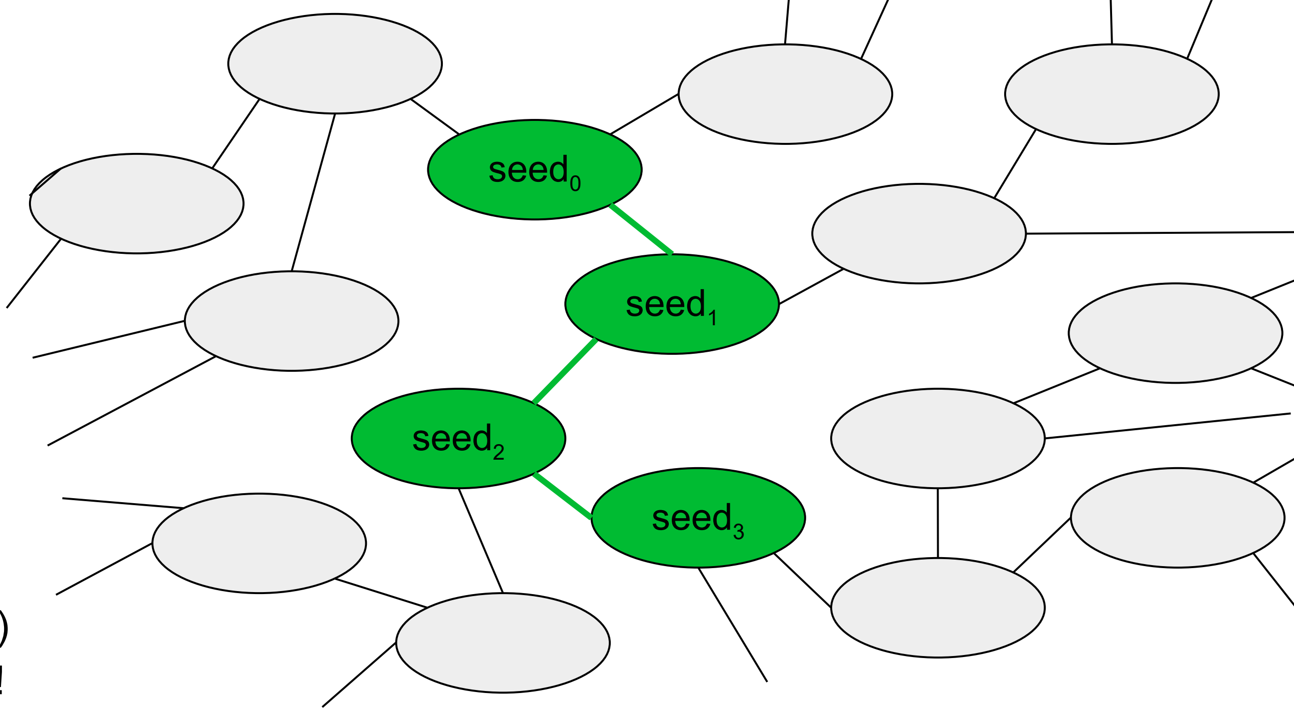
Unlike [GR20], select O(S) seeds using a length O(S) walk on expander graph.



Derandomization Step 2, Random Walk Set Sampling

Unlike [GR20], select O(S) seeds using a length O(S) walk on expander graph.

- Only takes seed length O(S) to sample length O(S) walk on length O(S) seeds!
- Walk is efficient, epsilon biased set is efficient, D is efficient.



$$walk = (seed_0, edge_1, edge_2, edge_3)$$

$$|walk| = O(S) + O(1) + O(1) + O(1)$$

Derandomization Step 2, Random Walk Set Sampling

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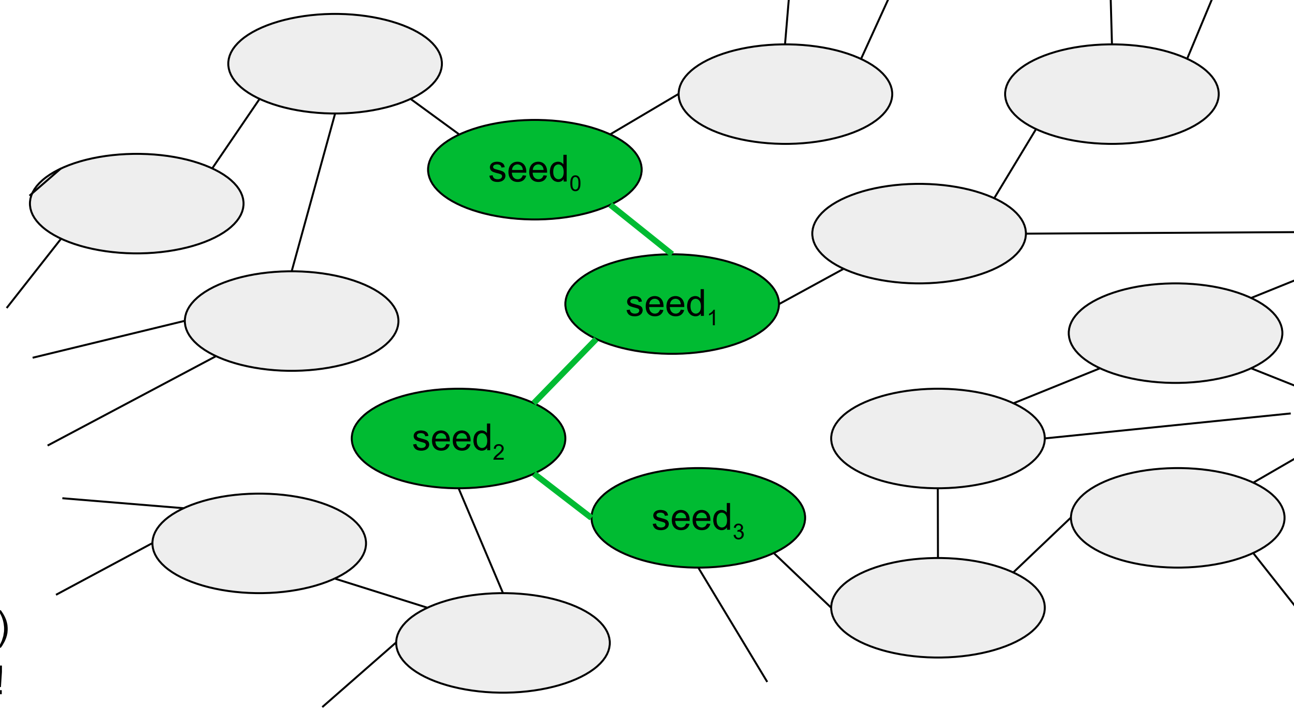

$$\text{walk} = (\text{seed}_0, \text{edge}_1, \text{edge}_2, \text{edge}_3)$$

$$|\text{walk}| = O(S) + O(1) + O(1) + O(1)$$

$$1 - \prod_{i \in [O(S)]} \left(1 - \sum_{w \in \{0,1\}^S} D(\text{seed}_i, w)\, M(u, w)\, M(w, v)\right)$$

Main Reduction

$$1 - \prod_{i\in\{0,1\}^\ell} (1 - \sum_{w\in\{0,1\}^S} D(walk_i, w)\ M(u, w)\ M(w, v))$$

- Represent i as binary, so it has $\ell = O(\log(S))$ bits.
- Remove one variable in i at a time.
  - For each variable need to do relinearization: $S\ \tilde{O}(\log(S))$
  - Need to do a product reduction: $\tilde{O}(\log(S))$
- Total time: $S \log(S)\ \tilde{O}(\log(S))$

Left with a claim about the multilinear extension of $D \circ walk$ and **M**.

Final points

- Need to run reduction log(T) times and compute multilinear extension of computation graph. Total time:

$$O(n + S \log(T) \log(S))\tilde{O}(\log(S))$$

- Use IP for deterministic algorithms to verify claim about $D \circ$ walk.
- If seeds fail, can prove they fail.
- Same idea works for unbounded fan-in circuits.
  - Faster than GKR for very large fan in.
  - Less optimized prover.

Contrast with GR20

Similar:

- Optimized for unbounded fan-in circuits.
- Uses Razborov Smolensky.
- Uses same epsilon biased sets.
- Achieve perfect completeness when seed is bad in same way.

Different:

- Optimized for time, instead of rounds.
    - More rounds to do degree reduction.
    - Lower degree polynomials.
    - Only constant number of claims at once.
- Requires further derandomization using random walks on expanders.
- Uses fast Interactive Provers for deterministic algorithms instead of direct arithmetization.

Open Problems

- Better Doubly Efficient Interactive Proofs (fast provers and verifiers).
- Extend Results to Threshold Circuits (Our results give fast verifiers for $AC[\oplus]$ circuits).
- Give protocols for randomized algorithms with simultaneous:
  - Same verifier time
  - Perfect completeness
  - $2^{O(S)}$ prover time (as opposed to $2^{O(S\log(T))}$).

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