TEXAS
The University of Texas at Austin

WHAT STARTS HERE CHANGES THE WORLD

Tighter Circuit Lower Bounds for MA/1 With Efficient PCPs

Joshua Cook
Joint work with Dana Moshkovitz

Trust Can’t Buy Time***

An Alternate Title

Untrusted Advice Vs Trusted Advice

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Untrusted, Adaptive	NTIME[T]	MATIME[T]
Trusted, Unadaptive	SIZE[T]*	BPTIME[T]/T
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$\mathrm{ONTIME}[n^4] \overset{?}{\subseteq} \mathrm{SIZE}[n]$

Can trusted programs always run faster than untrusted programs?

Santhanam: $\forall k>1$: $\mathrm{MATIME}[n^{O(k)}]/1 \not\subseteq \mathrm{SIZE}[O(n^k)]$

Non Explicit, Unbounded Polynomial

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Win-Win if a is small

There exists randomized programs with one bit of trusted advice and a long, untrusted program advice that cannot be solved much faster with trusted advice.

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Untrusted Merlin
Randomized Arthur.

Many Questions and Answers.

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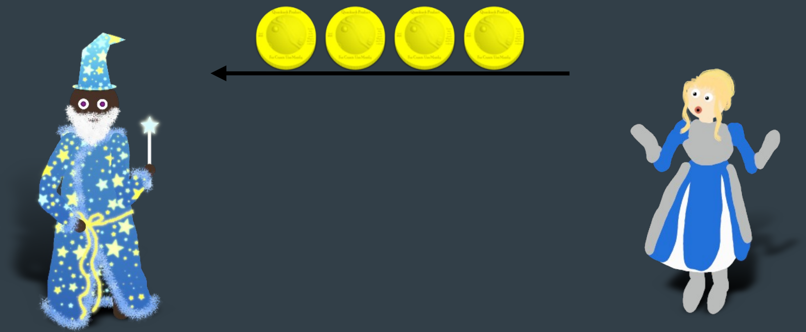

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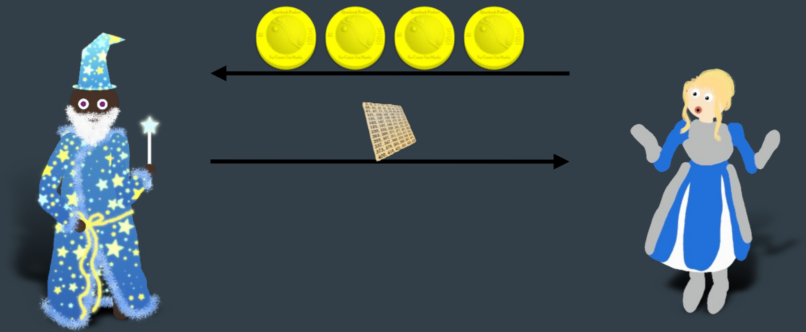

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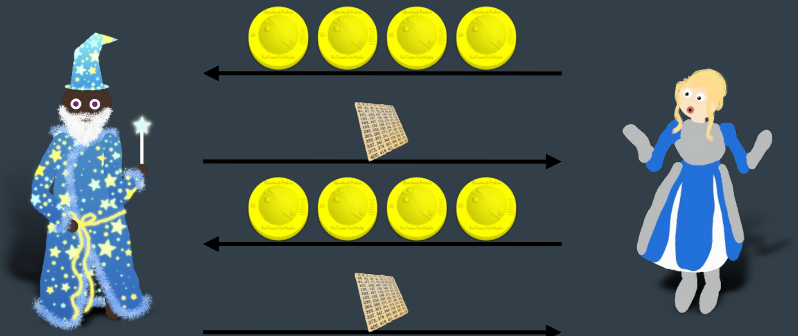

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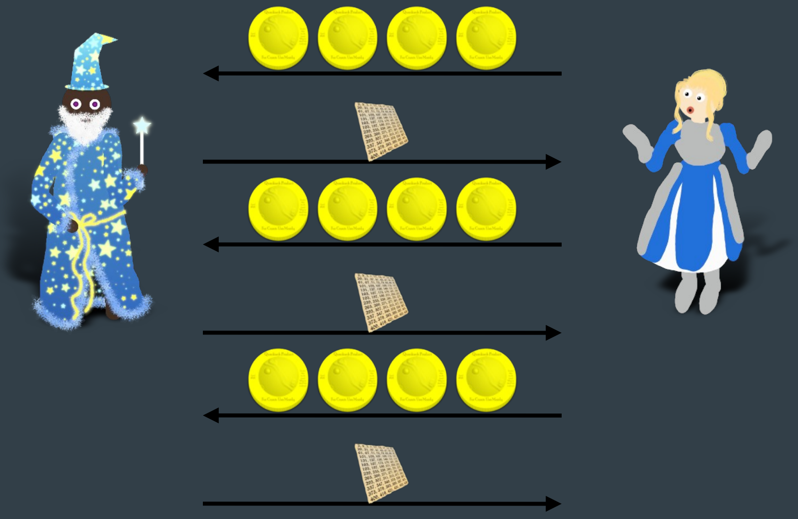

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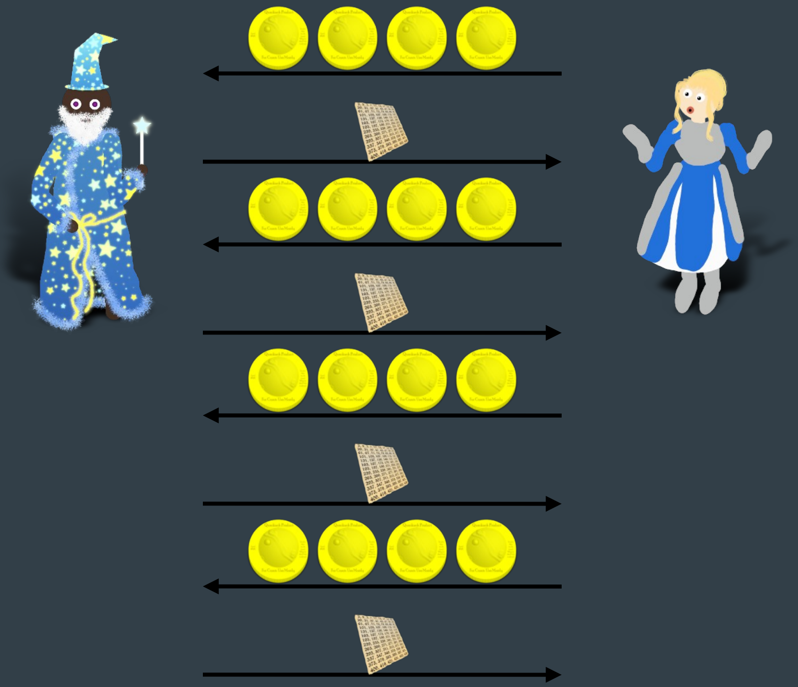

How powerful is IP?

Shamir 92 proved IP = PSPACE!

$$\mathrm{SPACE}[n] \subseteq \mathrm{IVTIME}[n^2]$$

$$\mathrm{IVTIME}[n] \subseteq \mathrm{SPACE}[n]$$

Prover's for IP also small space!

Circuit bounds for SPACE apply to IP!

Main Idea

Use a Circuit as Merlin in IP.

Merlin Gives a Circuit
Arthur Uses it to run IP

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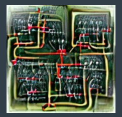

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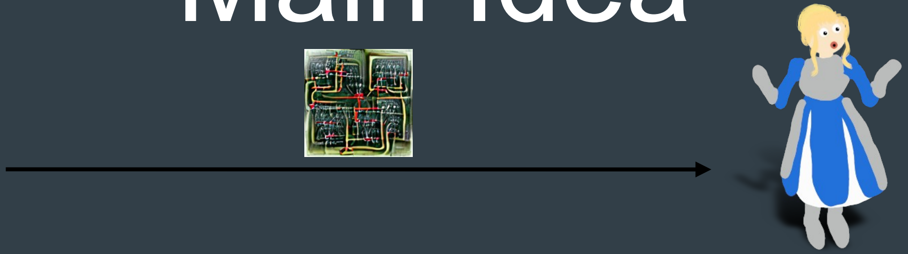

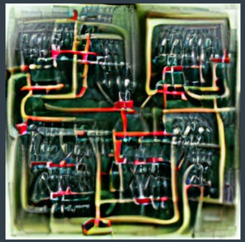

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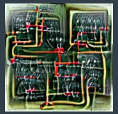

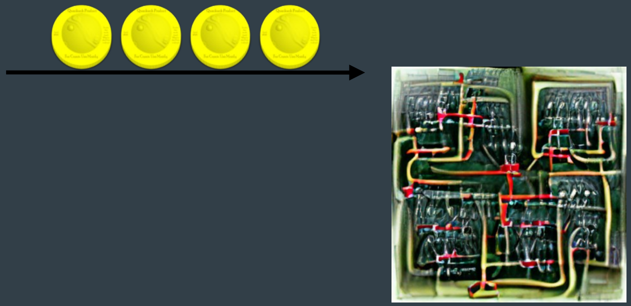

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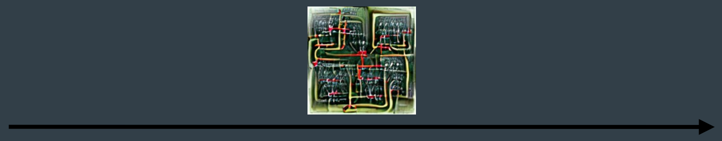

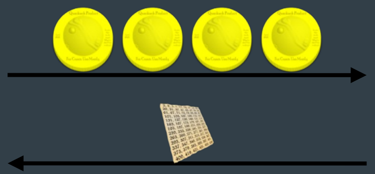

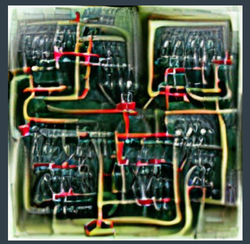

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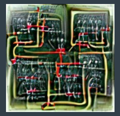

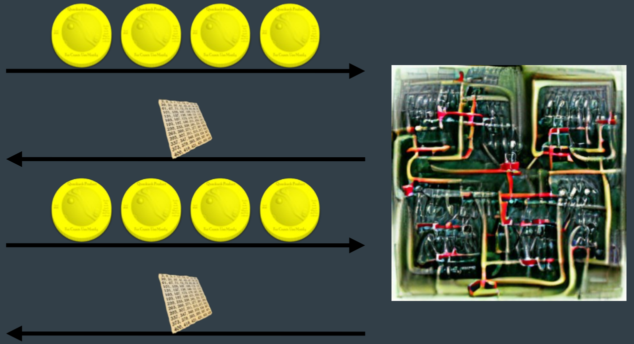

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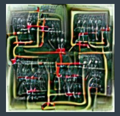

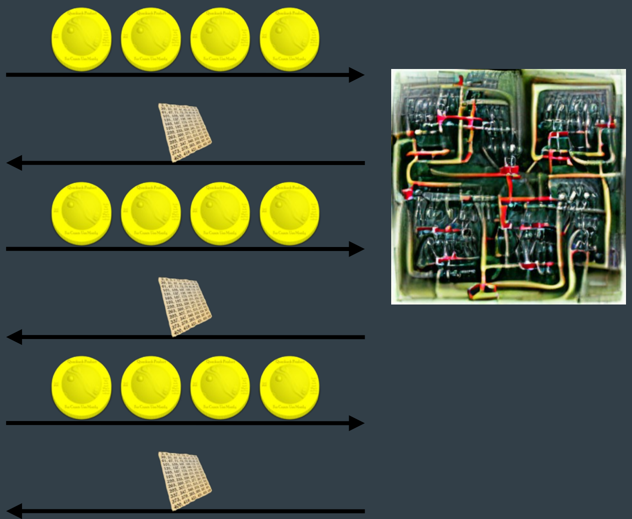

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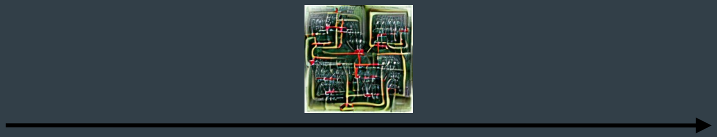

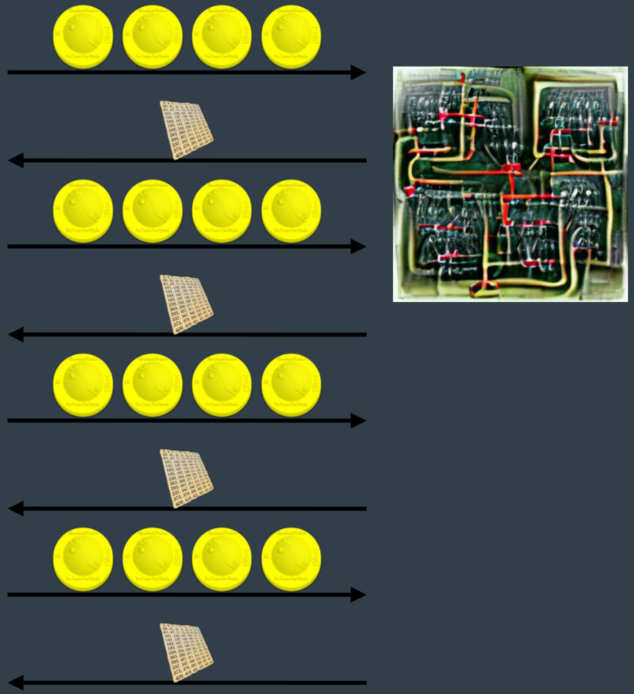

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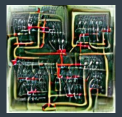

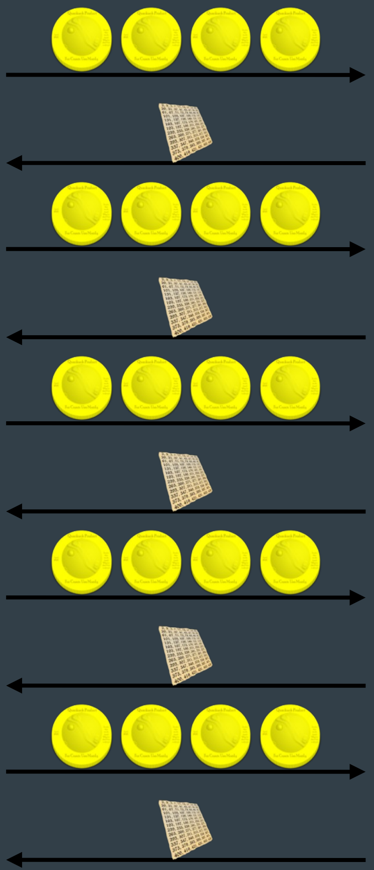

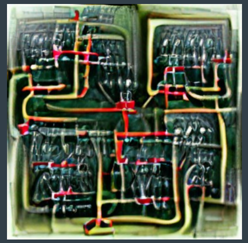

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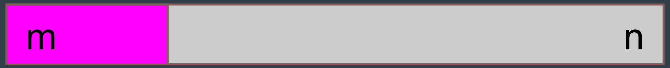

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- Idea: Use PCP to minimize verifier time, queries, interaction.

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New PCP Theorem

For Time-Space[T,S] there is PCP verifier with:

1. Verifier time O~(n+**logT**).
2. Prover space O~(S+n).
3. Queries O(logn + loglogT).
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Think of $T=2^n$ and $S=n$

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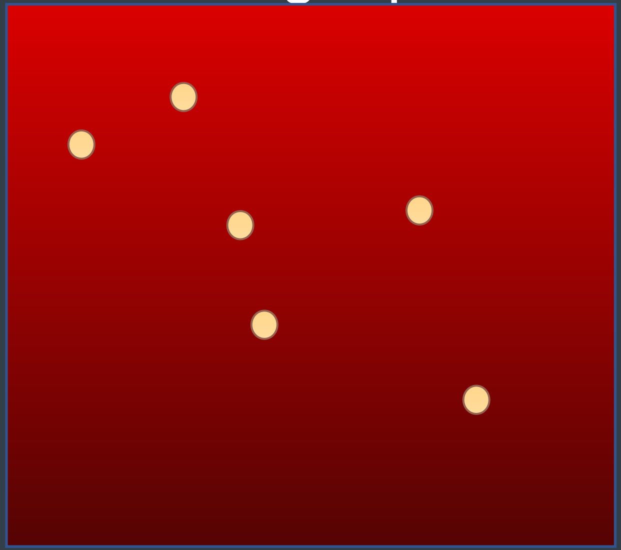
Holmgren-Rothblum`18 could give O~(n+logT) verifier time, but O(logT) queries

What Goes Into New PCP: Ultra-Efficient Query Reduction

- "Aggregation Through Curves": How to evaluate an m-variate low degree polynomial on k points using a prover?



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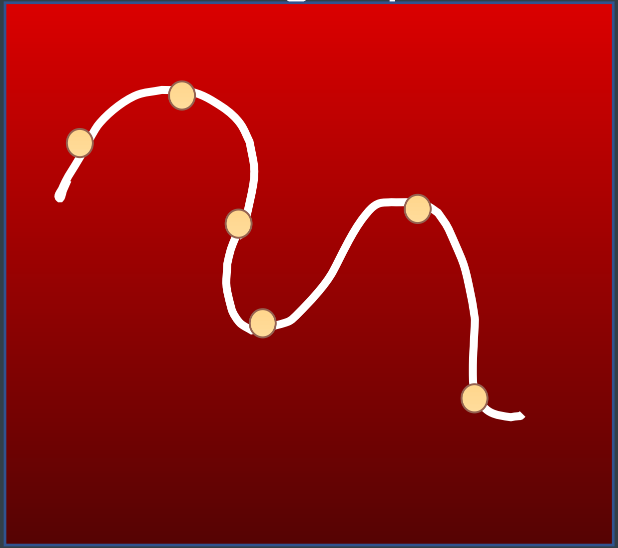

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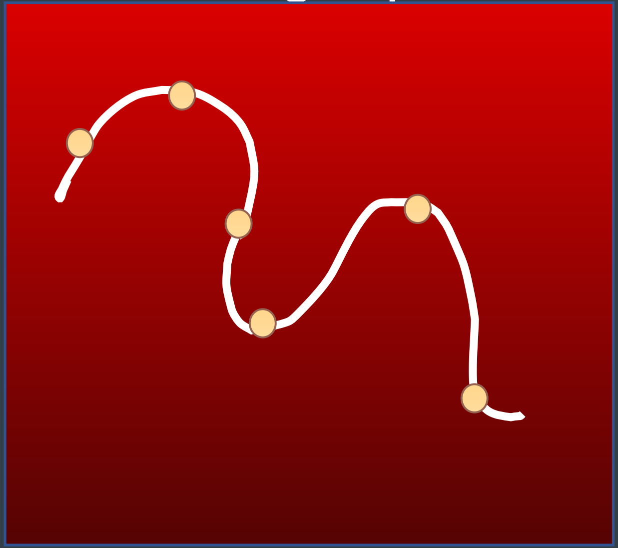

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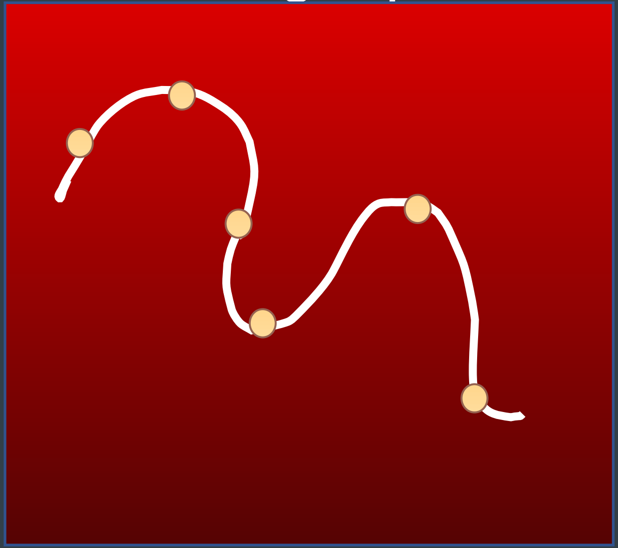

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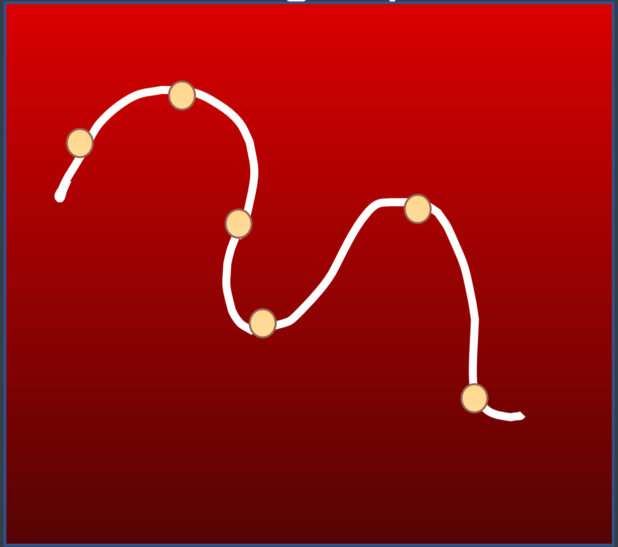

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Idea: need linear transformation of k points in time ~k+m. Possible for related points.

For Which k Prove $MATIME[n^{k+o(1)}]/1 \not\subset SIZE[n^k]$?

Have three cases:

1. $PSPACE \not\subset P/poly$
2. $SPACE[n] \subseteq SIZE[n^{1+o(1)}]$
3. **$\exists a>1$**: $SPACE[n] \subseteq SIZE[n^{a+o(1)}] - SIZE[n^{a-o(1)}]$

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All k. Santhanam’s IP works, part of input running IP on shrinks very quickly, poly overhead shrinks.

All k. Space ~ Size. From our PCP Space ~ Prover Space ~ Prover size.

For Which k Prove MATIME[$n^{k+o(1)}$]/1 $\not\subset$ SIZE[n^k]?

Have three cases:

1. PSPACE $\not\subset$ P/poly
2. SPACE[n] $\subseteq$ SIZE[$n^{1+o(1)}$]
3. $\exists$ **a>1**: SPACE[n] $\subseteq$ SIZE[$n^{a+o(1)}$] - SIZE[$n^{a-o(1)}$]

All k. Santhanam's IP works, part of input running IP on shrinks very quickly, poly overhead shrinks.

All k. Space ~ Size. From our PCP Space ~ Prover Space ~ Prover size.

k < a. For k = a, Space[n] $\not\subset$ Size [n^a], but Prover Space[n] ~ Size [$n^{a+o(1)}$]. So OMA time is about Size [$n^{a+o(1)}$]. Pad inputs for k < a.

For k > a, need something stronger than Space[n] for hard problem. Space hardness might stall, may need Space[n^k], but then prover requires Space [n^k], may need Size[n^{ka}].

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