Size Bounds on Low Depth Circuits for Promise Majority

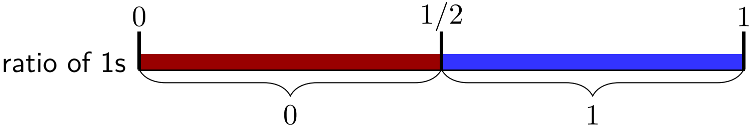
Joshua Cook

The University of Texas at Austin

July 3, 2022

Talk Outline

Result Overview

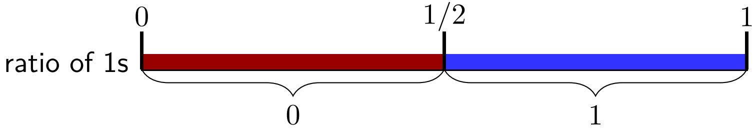


Definition (Majority)

For $n \in \mathbf{N}$, let Maj : $\{0,1\}^n \to \{0,1\}$ be defined by

$$\mathsf{Maj}(x) = \mathbf{1}\left[\sum_i x_i \geq n/2\right].$$

Majority



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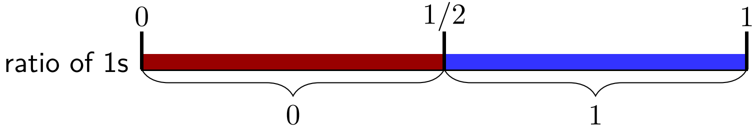

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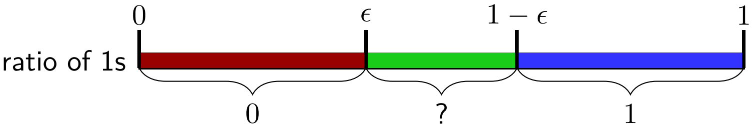
- Component of many results, such as circuit derandomization [1].
- Widely studied, not computable by AC0, simple computation models.

Promise Majority



Approximate majority[2], promise majority[6], approximate selector[4], etc.

Definition (Promise Majority)

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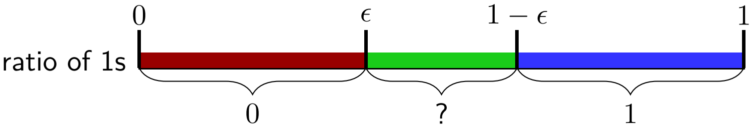

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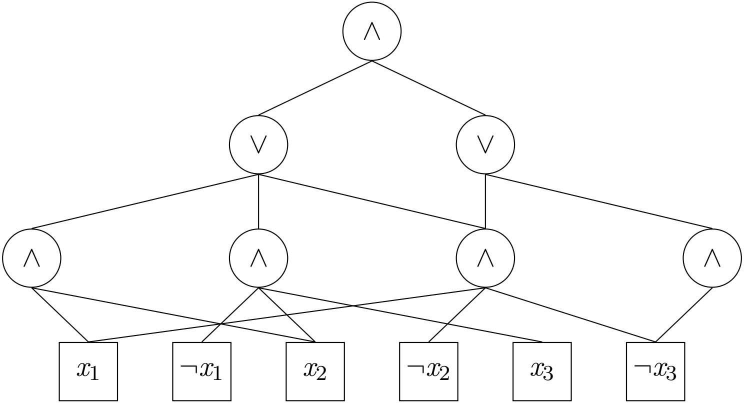

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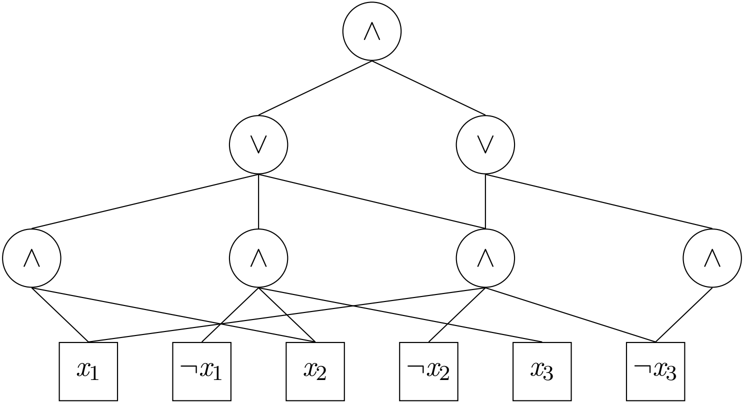

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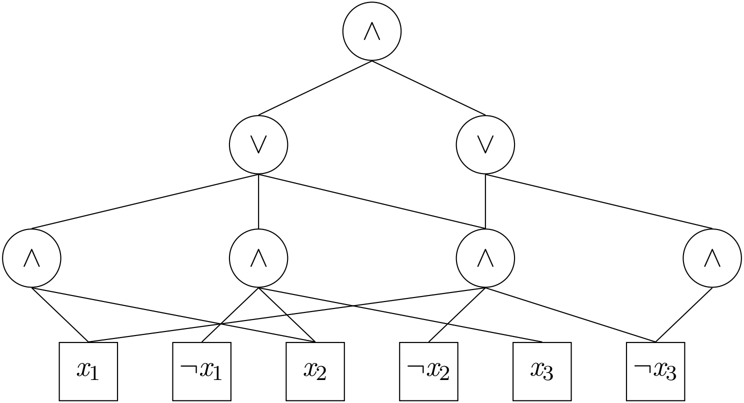

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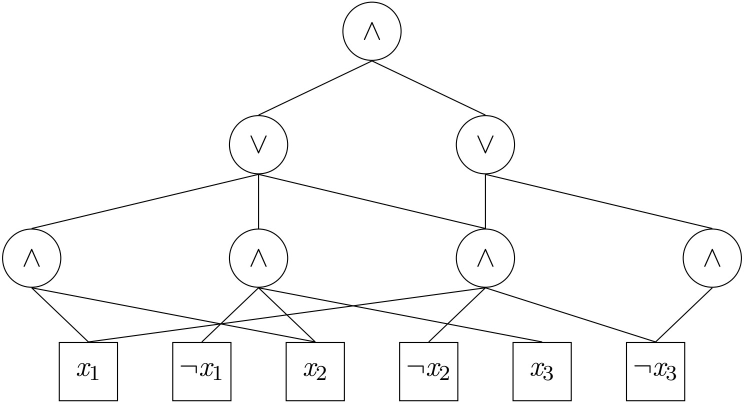

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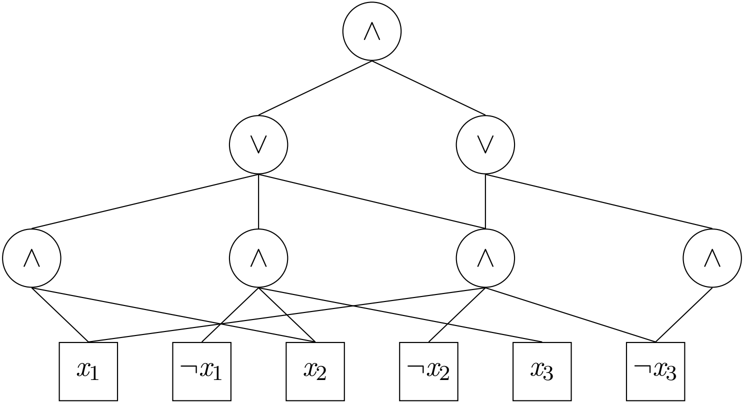

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- AC0 constant depth, polynomial size.

Depth-3 ϵ-Promise Circuit Upper Bounds

Depth-3 Upper Bounds:

Author	ϵ	Size	Uniformity
Ajtai 1983 [2]	$(0, 1/2)$	$(\epsilon \ln(\epsilon) n)^{2 + \frac{\ln(1-\epsilon)}{\ln(\epsilon) - \ln(1-\epsilon)}}$	Non-Uniform
Viola 2009 [7]	$\frac{1}{\ln(n)}$	$n^{4+o(1)}$	P
Viola 2009 [7]	$(0, 1/2)$	$n^{4+O((1-2\epsilon)^{-2})}$	P
Us	$\frac{1}{\ln(n)}$	$n^{3+o(1)}$	P

Depth-3 ϵ-Promise Circuit Lower Bounds

Depth-3 Lower Bounds (Suppressing polylogarithmic factors):

Author	Size	Monotone
Trivial	ϵn	General
Chaudhuri, Radhakrishnan 1996 [4]	$(\epsilon n)^{\frac{64}{63}} - n$	General
Viola 2011 [8]	$n^{\Omega(-\ln(1-2\epsilon))}$	General
Us	$\epsilon^3 n^{2+\frac{\ln(1-\epsilon)}{\ln(\epsilon)}}$	Monotone
Us	$\epsilon^3 n^{2+\frac{\ln(1-\epsilon^2)}{2\ln(\epsilon)}}$	General

Higher Depth ϵ-Promise Circuit Upper Bounds

Upper Bounds (Constant ϵ):

Author	Size	Uniformity
Ajtai 1990 [3]	$\text{poly}(n)$	LOGTIME
Chaudhuri, Radhakrishnan 1996 [4]	$n^{\frac{1}{1-2^{-O(d)}}}$	LOGTIME
Us	$n^{\frac{1}{1-2^{-(d-2)/2}}}$	Non-Uniform
Us	$n^{\frac{1}{1-(2/3)^{(d-2)/2}}}$	P

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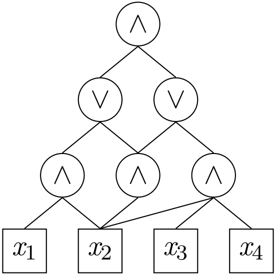
Chaudhuri, Radhakrishnan 1996 [4]	any	$(\epsilon n)^{\frac{1}{1-4^{-d}}} - n$
Viola 2011 [8]	$\frac{1}{2} - \frac{1}{\ln(n)^{d-2}}$	$\omega(\text{poly}(n))$

Depth 3 Circuits Terminology



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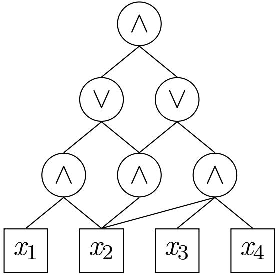

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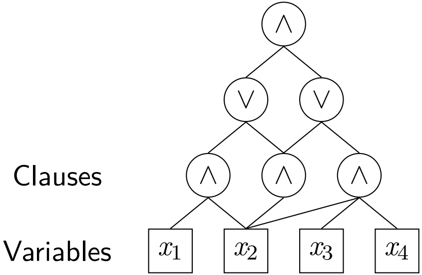

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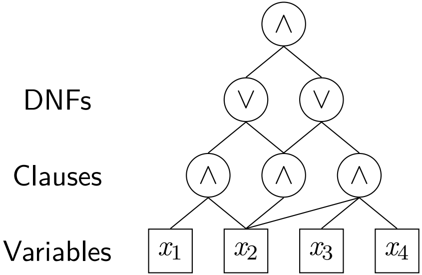

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- Second level, OR gates “DNFs”.

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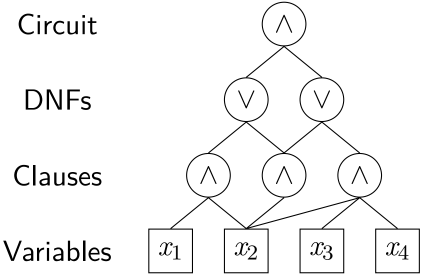

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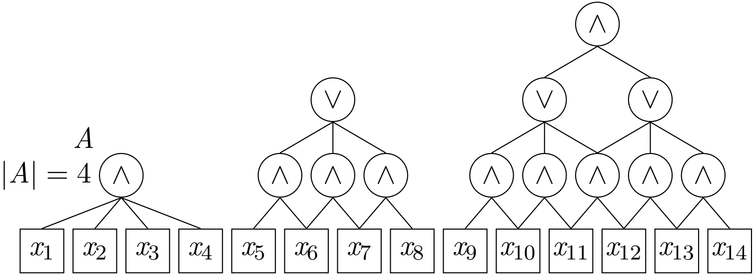
- First level, AND gates “clauses”.
- Second level, OR gates “DNFs”.
- Third level, AND gate “circuits”.

Size Definitions



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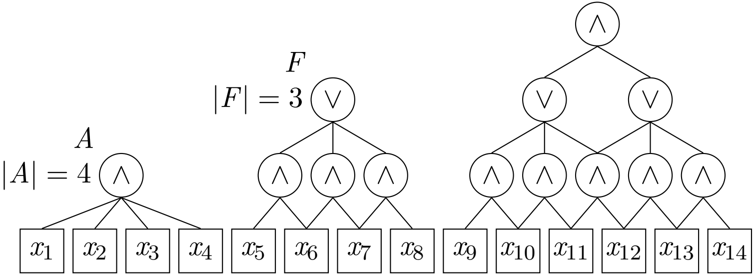

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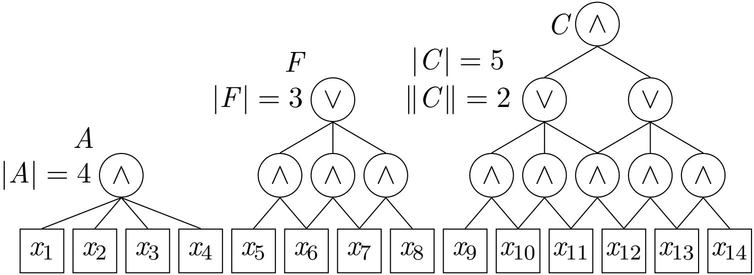

- Clause A, size $|A|$ is the number of variables in A.
- DNF F, size $|F|$ is the number of clauses in F.
- If C is a circuit, denote
  - $|C|$ as the number of clauses in C.
  - $\|C\|$ as the number of DNFs in C.
  - the size of C as $|C| + \|C\|$.

Monotone Lower Bound Idea

Idea: Lower bound the fan in at each layer.

Pretend $\epsilon \in (0, 1/2)$ is constant for simplicity. Let $\alpha = \frac{\ln(1-\epsilon)}{\ln(\epsilon)}$.

1. From Viola [7], clauses have size $\frac{\ln(n)}{\ln(1/\epsilon)}$.

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   Thus clauses have size $\tilde{\Omega}(n^{1+\alpha})$.
3. If fewer than $\tilde{o}(n^{2+\alpha})$ clauses, can hit every DNF with fewer than $\frac{n}{\ln(n)^2}$ clauses.
   Thus circuit has $\tilde{\Omega}(n^{2+\alpha})$ clauses.

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Solution: Let $\beta = \frac{\ln(1-\epsilon^2)}{2\ln(\epsilon)}$. Fix adversarial bits *probabilistically*.

Result: Likely won't set DNF to one.
Almost definitely will eliminate $\tilde{\Omega}(n^{1+\beta})$ clauses.

Main Tool

Greedy Algorithm for set cover.

Theorem

Let $S = \{S_1, ..., S_m\}$ be subsets of $[n]$ where each $i \in [m]$ has $|S_i| \geq w$. Then for any $t \in [n]$ there is some $T \subseteq [n]$ with $|T| = t$ so that T doesn't intersect with at most

$$me^{w \ln(1 - \frac{t}{n+1})}$$

of the sets in S.

Idea: Just greedily take the variable in the most sets.

Upper Bound

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3. Repeat with appropriate group d times.

Circuit has depth $2+2d$ and size $\tilde{\Omega}\left(n^{\frac{1}{1-2^{-d}}}\right)$.

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We use this circuit to get small uniform upper bounds with more depth.

Monotone Depth-3 Lower Bound

What We Actually Show

Here we prove the simpler lower bounds for constant $\epsilon \in (0, 1/2)$ of:

Monotone

$$n^{2+\Omega\left(\frac{\epsilon}{\ln(1/\epsilon)}\right)}$$

General

$$n^{2+\Omega\left(\frac{\epsilon^2}{\ln(1/\epsilon)}\right)}$$

The tighter bounds follow the same ideas with tighter analysis.

Biased Coin Distributions

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By central limit theorem, with probability almost one half, D_ϵ has less than ϵ fraction ones.

Restriction

Definition

We say $\rho \in \{0, 1, *\}^n$ is a restriction on n bits. We say the size of ρ, $|\rho|$, is the number of 1s and 0s in ρ.
If $f : \{0, 1\}^n \to \{0, 1\}$, then define $f \restriction_\rho$ as the function where the values from ρ are passed into f where it is 1 or 0, and otherwise the corresponding variable from the argument is passed in.

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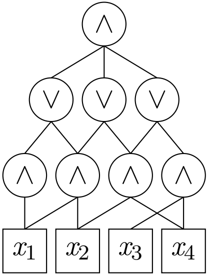

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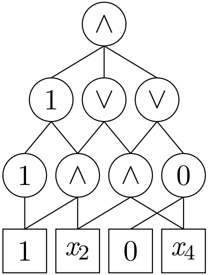

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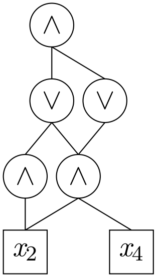

Clause Lower Bound

Viola proved:

Theorem

Suppose that for constant $\epsilon \in (0, 1/2)$, and DNF F that

$$\Pr[F(D_\epsilon) = 0] \geq \textit{poly}(1/n).$$

Then for some $w = \Omega(\frac{\ln(n)}{\ln(1/\epsilon)})$, there is a restriction ρ restricting at most $m = \frac{\epsilon n}{\ln(n)}$ variables so that:

- *Any clause C in F with width less than w has $C \restriction_\rho = 0$.*
- $\Pr[F \restriction_\rho (D_\epsilon) = 0] \geq \Pr[F(D_\epsilon) = 0]$

Eliminates small clauses from a DNF that is likely to output a 0 on D_ϵ with few variables without setting the DNF to 1.

Viola's Clause Lower Bound Idea

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- Independent $T \implies$ set cover requires size $|T|$.

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Large Independence: Not possible! Small width on D_ϵ outputs 1 too often.

Small Independence: Fix few variables in small cover to reduce width.

- Choose values to only increase probability of 0.
- Repeat until clause width 0.

Greedy Set Cover

In this talk, we use

Theorem

Let $S = \{S_1, ..., S_m\}$ be subsets of $[n]$ where each $i \in [m]$ has $|S_i| \geq w$. Then for any $t \in [n]$ there is some $T \subseteq [n]$ with $|T| = t$ so that T intersects all but at most

$$|S|e^{-w\frac{t}{n}}$$

of the sets in S.

Closer analysis gives that T intersects all but $|S|e^{-w\ln(1-\frac{t}{n+1})}$ sets.

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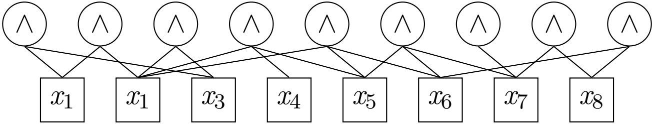
Closer analysis gives that T intersects all but $|S|e^{-w\ln(1-\frac{t}{n+1})}$ sets.
In particular, if

- S is the set of clauses in a monotone DNF, F, and
- ρ is some restriction restricting variables in T to 0,

then $|F\restriction_\rho| \leq |F|e^{-w\frac{t}{n}}$ variables remaining.

Greedy Set Cover Proof



- The average number of sets an element is in is at least $\frac{w|S|}{n}$. So at least one variable, say x_2, is in at least $\frac{w|S|}{n}$ sets.

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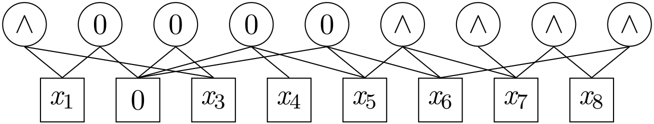

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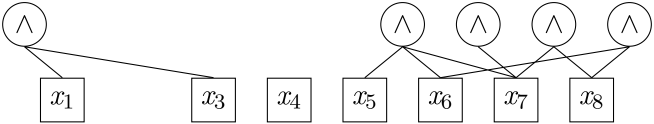

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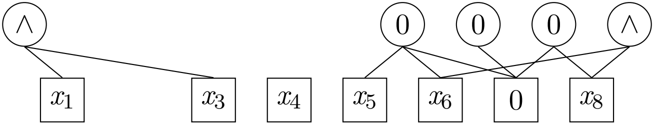

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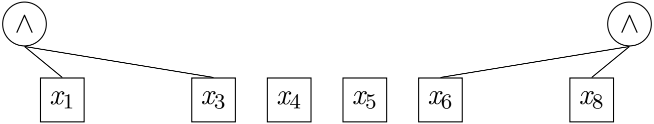

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- Then these t variables work.

Monotone DNF Size

Theorem

Let $\epsilon \in (0, 1/2)$ *and monotone DNF* F *be suchthat*

- *For all x with less than* ϵn *zeros,* $F(x) = 1$.
- $\Pr[F(D_\epsilon) = 0] \geq$ *poly*$(1/n)$.

Then F has $n^{1+\alpha}$ *clauses for some* $\alpha = \Omega(\frac{\epsilon}{\ln(1/\epsilon)})$.

All DNFs in circuit must satisfy condition 1.

But For DNF to "help" by much, it must satisfy condition 2.

Monotone DNF Size Proof

1. Using Viola's theorem, we fix $\frac{n}{\ln(n)}$ variables and F is left with only clauses longer larger than $w = \Omega(\frac{\ln(n)}{\ln(1/\epsilon)})$.

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4. Together

$$\begin{aligned} \frac{\epsilon n}{3} &\leq |F| n^{-\Omega\left(\frac{\epsilon}{\ln(1/\epsilon)}\right)} \\ n^{1+\alpha} &\leq |F|. \end{aligned}$$

Monotone Circuit Size Lower Bounds

Theorem

Depth-3 Circuit C solving ϵ-promise majority has size $n^{2+\Omega(\frac{\epsilon}{\ln(1/\epsilon)})}$.

Idea: Eliminate many DNFs with few clauses.

Can eliminate too many DNFs if there are not enough clauses.

Eliminate All Large Clauses

Theorem

For any Circuit C with $|C| \leq n^c$*, there is a restriction* ρ *restricting* $c\frac{n}{\ln(n)}$ *variables such that* $C \restriction_\rho$ *has no clauses larger than* $\ln(n)^2$*.*

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Eliminate with Greedy Cover Algorithm! Fix $c\frac{n}{\ln(n)}$ variables with restriction ρ so that

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NOTE: Similar algorithm works if the clauses are non-monotone, but must generalize theorem.

Monotone Lower Bound Final Ingredient

Simple version of final step in circuit lower bound.

Theorem

If F is a monotone DNF with clause width m^{1+x} for constant $x > 0$, $|F| =$ poly(n) and such that F computes "OR", then F must have $n \geq \tilde{\Omega}(m^{2+x})$.

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2. See that $m < m^{1+x} \leq n$. So $F \restriction_\rho \neq 0$, and $|F \restriction_\rho| \geq 1$.
3. Together:
$$\begin{aligned} 1 \leq& |F| e^{-m^{2+x} \frac{1}{n}} \\ \tilde{\Omega}(m^{2+x}) \leq& n \end{aligned}$$

Monoton Circuit Lower Bound Proof Idea

- Remove Large Clauses.
- Use DNF lower bounds to get each cause bigger than $n^{1+\alpha}$.
- Fix whole clauses with the idea from the previous slide to lower bound number of clauses.

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Issue: Some DNFs might be small.

Solution: Focus on large DNFs during elimination.

Insight: Some large DNF must survive if few variables fixed.

Monotone Circuit Lower Bound Proof

Let C be a circuit solving ϵ-promise majority.

- Remove all clauses larger than $\ln(n)^2$ with a restriction ρ_1 which restricts $O\left(\frac{n}{\ln(n)}\right)$ variables.

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- See that $C\restriction_{\rho_1}\restriction_{\rho_2}$ still solves $\left(\epsilon - O\left(\frac{1}{\ln(n)}\right)\right)$-promise majority. If $\|C\| \leq n^3$, by a counting argument some DNF, F, must have $\Pr[F(D_\epsilon) = 0] \geq \text{poly}(1/n)$. Thus, $\|C'\restriction_{\rho_2}\| \geq 1$.

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- Together:

$$\tilde{\Omega}(n^{2+\alpha}) = n^{2+\Omega\left(\frac{\epsilon}{\ln(1/\epsilon)}\right)} \leq |C'|.$$

General Depth-3 Lower Bounds

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  Will fix next.
- Circuit lower bounds, works!
  - At worst, might eliminate or shrink DNFs and clauses early.
  - But circuit still solves a promise problem, so it still has large DNFs after restriction.

Probabilistic Restriction

Main Lemma

Lemma

For constant $\epsilon \in (0, 1/2)$, let F be a DNF with:

- *For all x with less than ϵn zeros, $F(x) = 1$.*
- $\Pr[F(D_\epsilon) = 0] \geq$ *poly*$(1/n)$

Let $\beta = \Omega(\frac{\epsilon^2}{\ln(1/\epsilon)})$. Then there is a random variable ρ which is a restriction on $\epsilon n/2$ variables such that:

- $F(D_\epsilon) = F \restriction_\rho (D_\epsilon)$.
- *Let F' be the DNF with clauses in $F \restriction_\rho$ bigger than $w = \Omega\left(\frac{\ln(n)}{\ln(1/\epsilon)}\right)$. Then: $\Pr[|F'| > |F|n^{-\beta}] \leq e^{-\Omega(n)}$.*

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Use greedy set cover algorithm to choose variables like monotone case.

Instead of just setting them to 0, we set them to 1 with probability ϵ.

Then by Chernoff bounds, its likely that we eliminate many clauses.

And by definition if we restrict the rest of the variables, it is the same as using D_ϵ.

Probabilistic Restriction Construction

First, define sequence of DNFs $F_1, ..., F_m$, and restrictions $\rho_0, ..., \rho_m$ for $m = \epsilon n/2$.

1. Let F_1 be the DNF only including clauses from F with width larger than $w = \Omega\left(\frac{\ln(n)}{\ln(1/\epsilon)}\right)$ from the clause lower bound.
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Then $\rho = \rho_m$, and F' is the DNF with clauses from $F_m \restriction_\rho$ bigger than w. See that $F \restriction_{\rho_m} (D_\epsilon) = F(D_\epsilon)$.

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If k steps succeed, then

$$|F_m \restriction_{\rho_m}| \leq (1 - \frac{w}{2n})^k |F| \leq |F| e^{-\frac{wk}{2n}} \leq |F| n^{-\Omega\left(\frac{k}{\ln(1/\epsilon) n}\right)}.$$

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Thus

$$\Pr[|F_m \restriction_{\rho_m}| > |F| n^{-\Omega\left(\frac{\epsilon m}{\ln(1/\epsilon)n}\right)}] \leq e^{-\Omega(n)}$$
$$\Pr[|F'| > |F| n^{-\Omega\left(\frac{\epsilon^2}{\ln(1/\epsilon)}\right)}] \leq e^{-\Omega(n)}.$$

Applying Restriction To Get DNF Bounds

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4. $F \restriction_{\rho'}$ has $\Omega(\epsilon n)$ clauses with width $\Omega\left(\frac{\ln(n)}{\ln(1/\epsilon)}\right)$: $|F'| \geq \Omega(\epsilon n)$. Thus:
$$\begin{aligned} \Omega(\epsilon n) \leq |F'| &\leq |F|n^{-\beta} \\ n^{1+\beta} &\leq |F|. \end{aligned}$$

General Circuit Lower Bounds

Use the same argument as the monotone DNF, with the lower bounds of $n^{1+\beta}$ on the second level.

Upper Bounds

Depth-3 Upper Bounds

Upper bounds use depth-3 circuits as a subroutine.

For constant ϵ, we use:

Existential: constant ϵ: Ajtai gave size $O\left(n^{2+\frac{\ln(1-\epsilon)}{\ln(\epsilon)-\ln(1-\epsilon)}}\right)$.

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Reminder, Idea: Amplify, recursively apply promise majority.

Amplification

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  Results $o(n^2)$-sized depth-4 circuits using careful analysis of Ajtai's.

- Can we get very small promise majority with just amplification and a single depth-3 promise majority?
  - Not without much better amplification!
  - Existing techniques increase size faster than promise, so that depth-3 promise majority circuits solving promise majority are still 'large'.

Iteratively Computing Majority Idea

$\epsilon = 2/5$ $\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$

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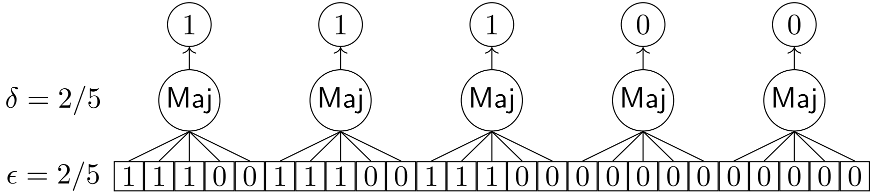
Iterative Computing Majority Idea



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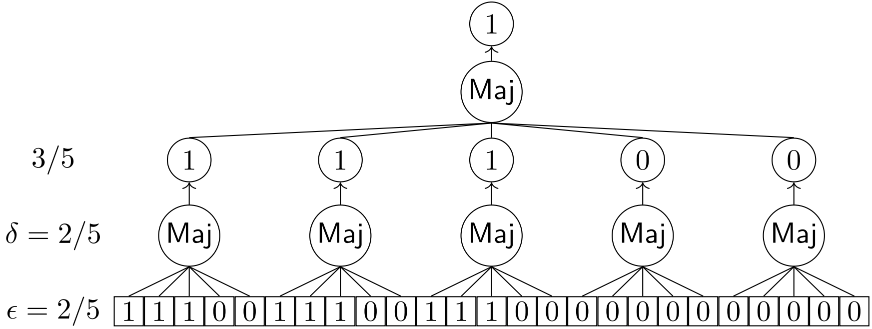

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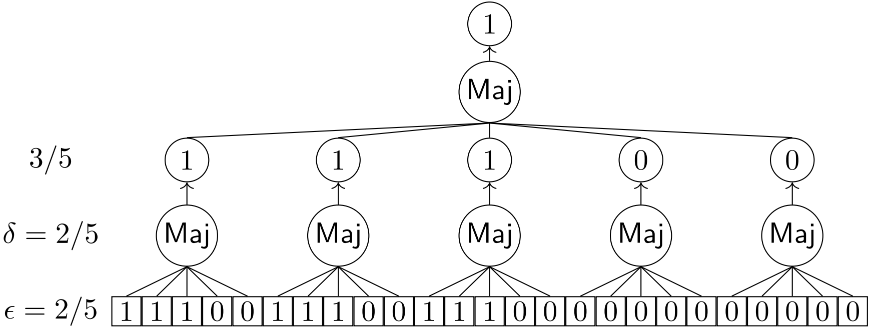

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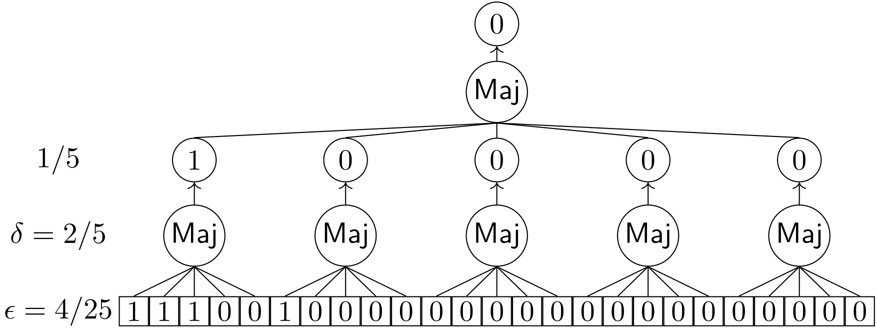

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Solution: Amplify, then run in groups.

Iterately Computing Majority

Using this idea:

Theorem

If there are depth-3 circuits with size n^α solving $\frac{1}{\ln(n)}$-promise majority, then for any positive integer k, there are depth-$(1+2k)$ circuits solving $\frac{1}{\ln(n)^k}$-promise majority with size

$$kn^{\frac{1}{1-\left(\frac{\alpha-1}{\alpha}\right)^k}}.$$

Combined with depth 2 amplification, we get our upper bounds for higher depths.

Sub-Quadratic Size Promise Majority

As special cases, we get, using Ajtai's circuit, we get:

Theorem

There exists a depth-4 circuits computing ϵ-promise majority with size $o(n^2)$.

And using our circuit, we get:

Theorem

There exists a depth-6, P-Uniform circuits computing ϵ-promise majority with size $o(n^2)$.

Viola's original circuit needed depth-8 to get sub-quadratic size.

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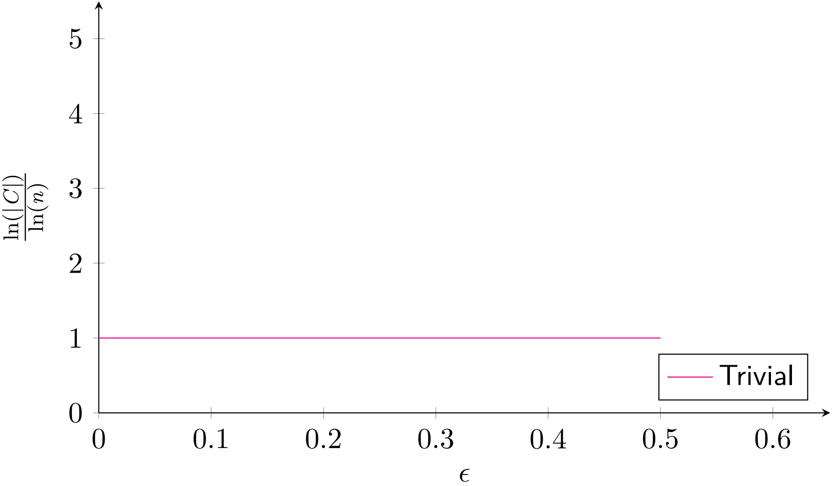
- Missing explicit, depth-4 quadratic sized circuits.
  - Seems related to other psuedorandom objects. Can be rephrased as distribution over dispersers.
- Results aren't tight!
  - Upper and lower bounds don't match.
  - Are the best circuits monotone?
  - Do any uniform circuits have optimal size?
  - Upper bounds for large depth don't match known lower bounds (Chaudhuri and Radhakrishnan are asymptotically close [4]).

Depth-3 Bounds, Constant ϵ

Note: Graphs *slightly* adjusted for visibility.
Y-axis is c for circuit size n^c.



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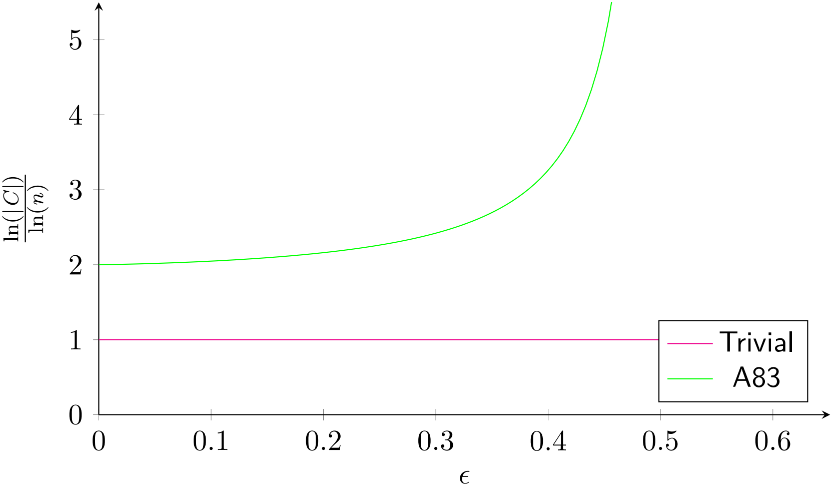

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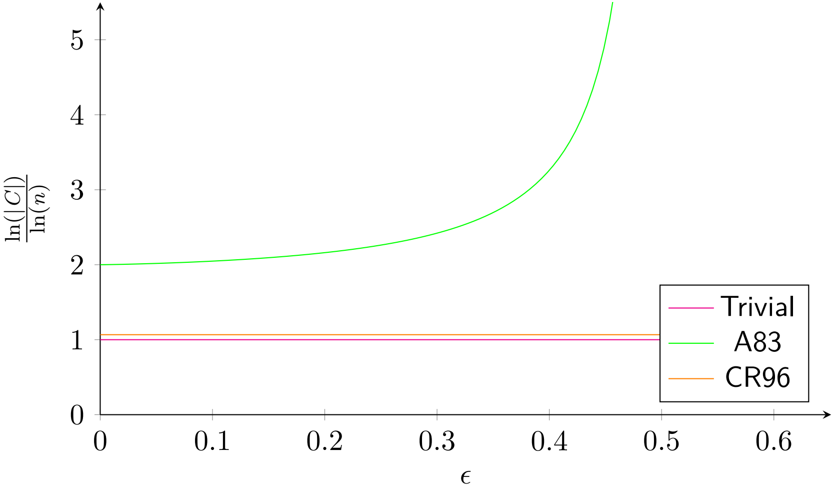

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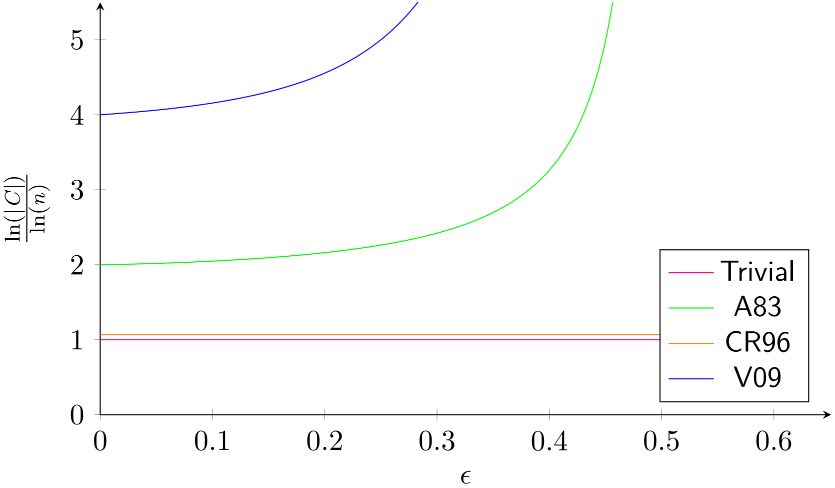

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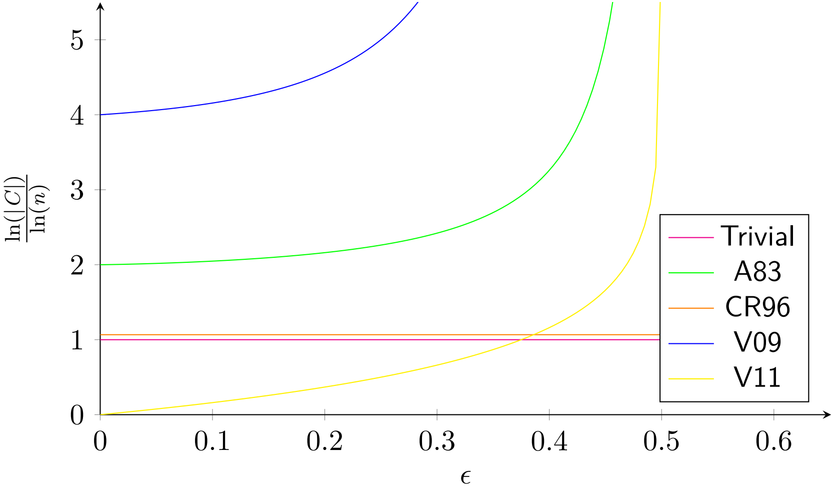

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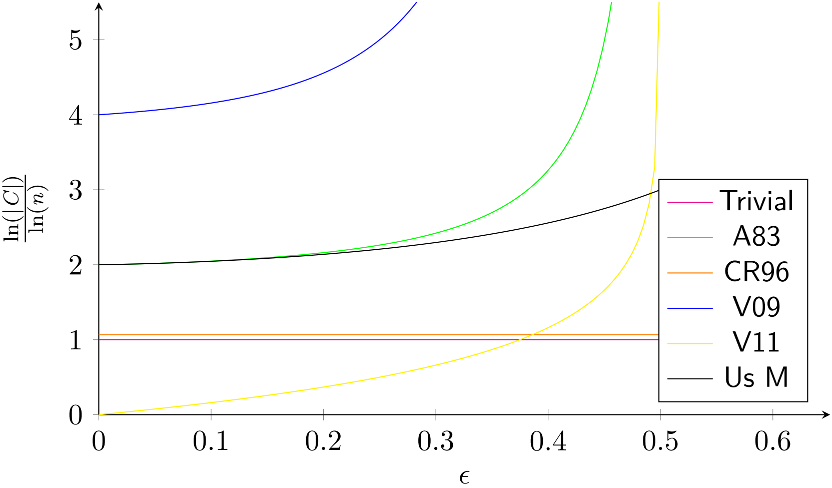

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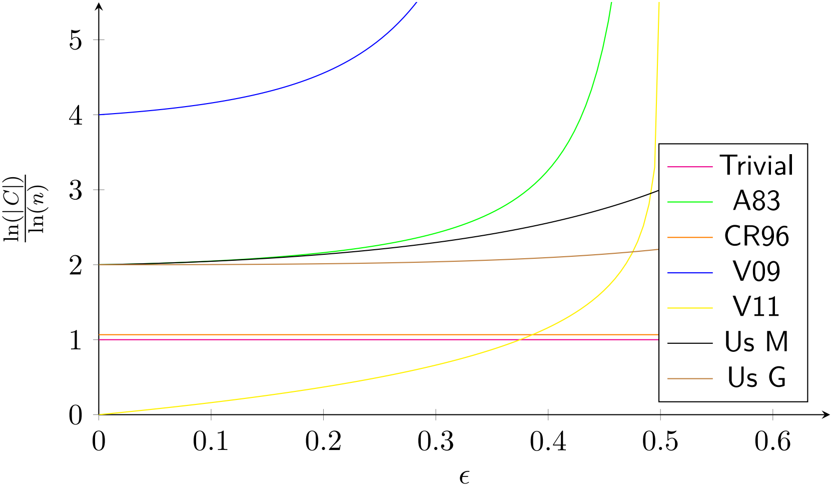

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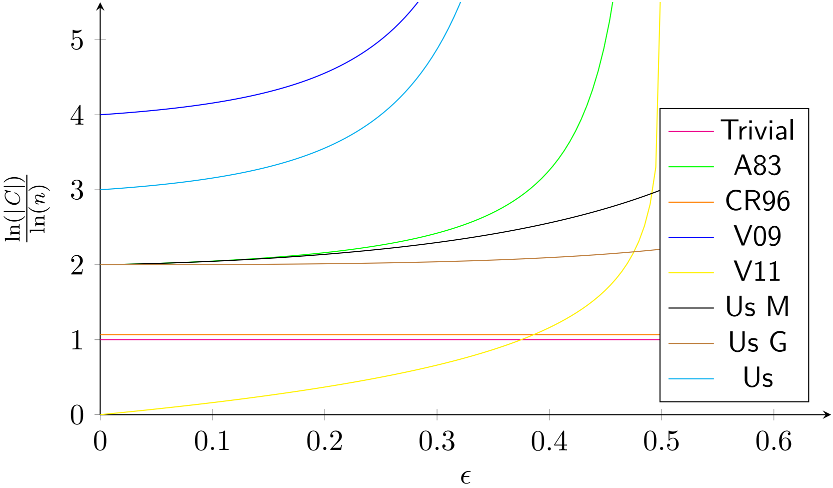
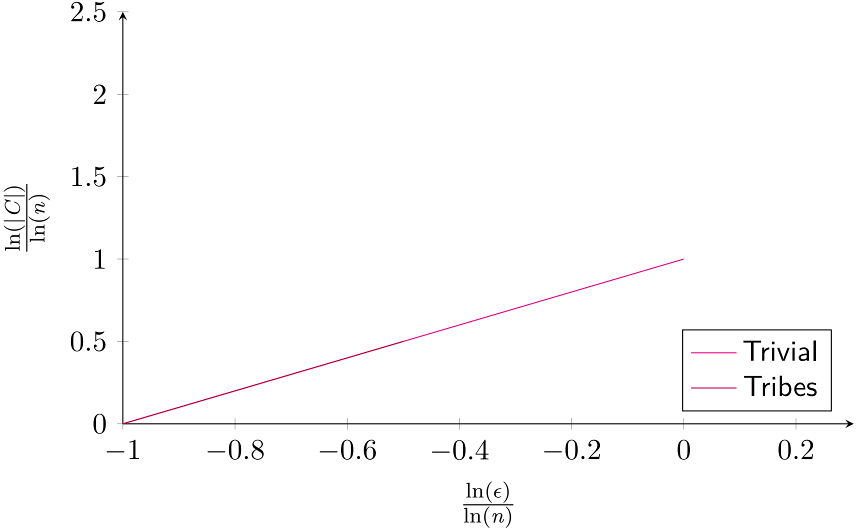


Depth-3 Bounds, Small ϵ

Note: *now* might have many more wires than gates.

X-axis is c if $\epsilon = n^c$.



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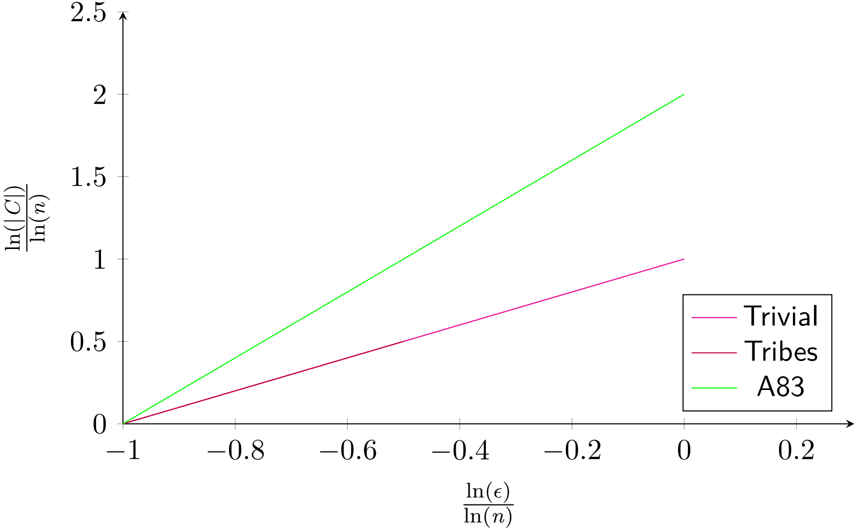

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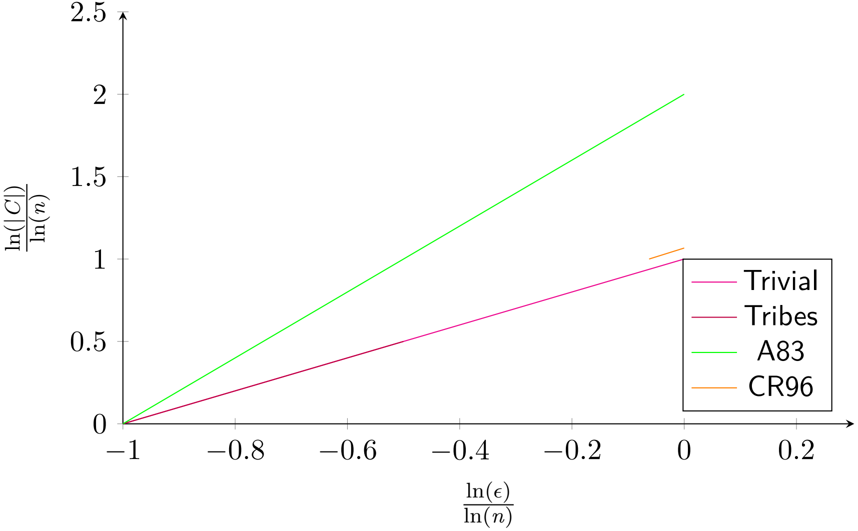

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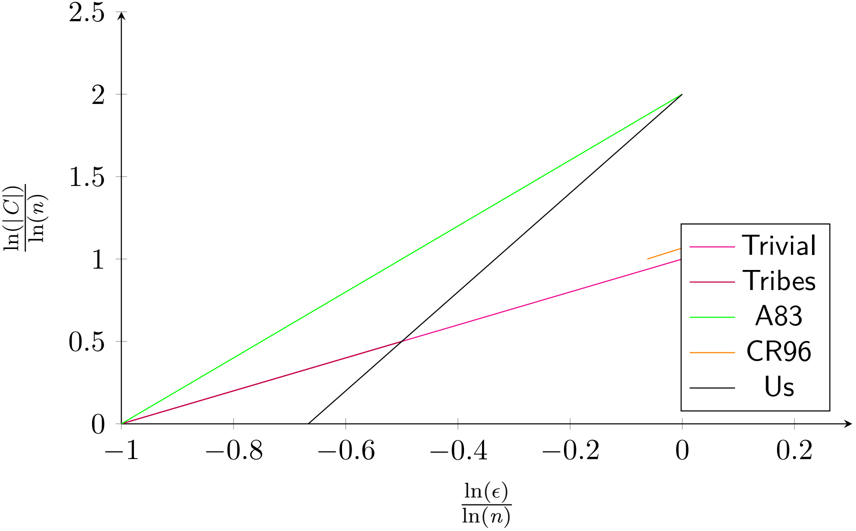

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