Size Bounds on Low Depth Circuits for Promise Majority

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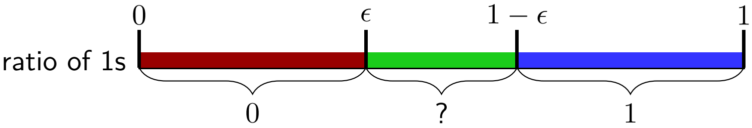
Promise Majority



Approximate majority[1], promise majority[3], approximate selector[2], etc.

Definition (Promise Majority)

For $n \in \mathbf{N}$, $\epsilon \in (0, 1/2)$, and function $f : \{0,1\}^n \to \{0,1\}$, we say f solves ϵ-promise majority if for all $x \in \{0,1\}^n$ with $\sum_{i\in[n]} x_i < \epsilon n$ and for all $y \in \{0,1\}^n$ with $\sum_{i\in[n]} 1 - y_i < \epsilon n$

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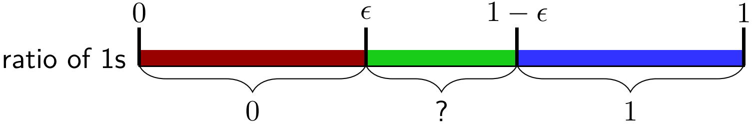

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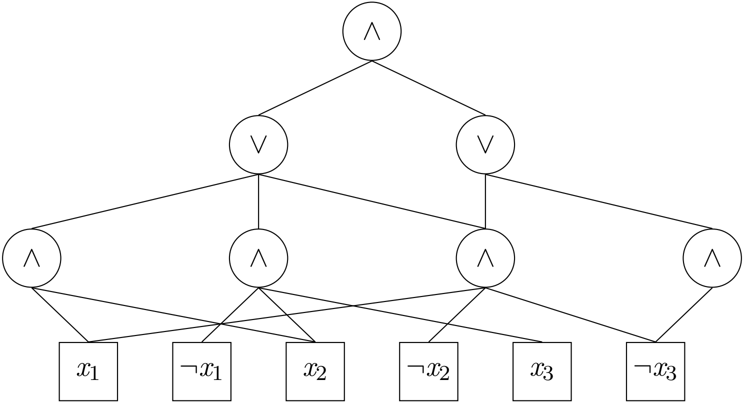
- Often usable in place of majority, in circuit derandomization.
- Widely studied, computable by AC0.

AC0



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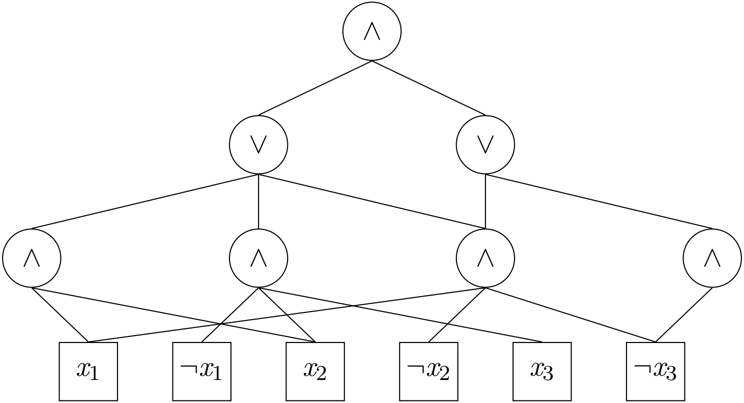
AC0



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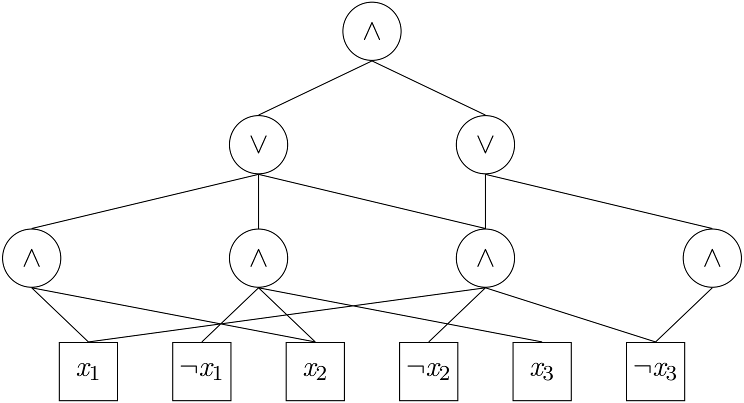

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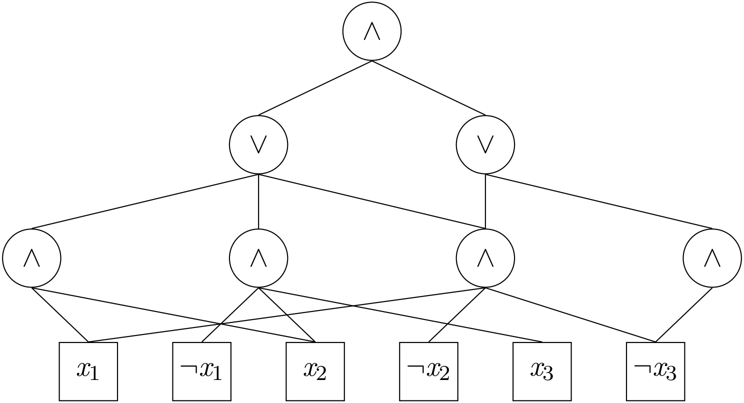

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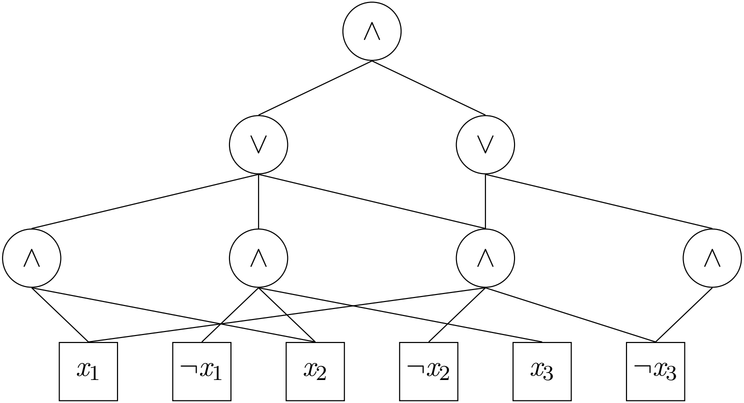

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- Size is number of gates (same results for wires).
- AC0 constant depth, polynomial size.

Depth-3 ϵ-Promise Circuit Bounds

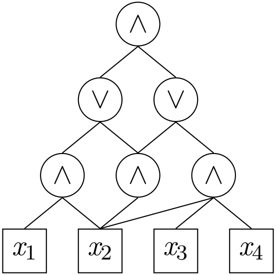
Depth-3 Lower Bounds (Suppressing polylogarithmic factors):

Author	Size	Monotone
Trivial	n	General
Chaudhuri, Radhakrishnan 1996 [2]	$n^{\frac{64}{63}}$	General
Viola 2011 [5]	$n^{\Omega(-\ln(1-2\epsilon))}$	General
Us	$n^{2+\frac{\ln(1-\epsilon)}{\ln(\epsilon)}}$	Monotone
Us	$n^{2+\frac{\ln(1-\epsilon^2)}{2\ln(\epsilon)}}$	General

Circuit Upper Bound by Ajtai 1983 [1]:

$$n^{2+\frac{\ln(1-\epsilon)}{\ln(\epsilon)-\ln(1-\epsilon)}}.$$

Depth-3 Circuits Terminology



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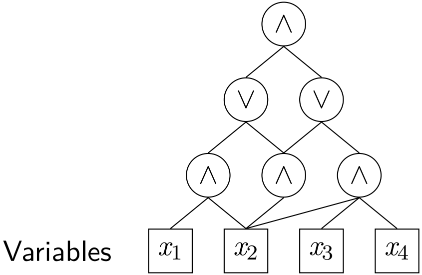

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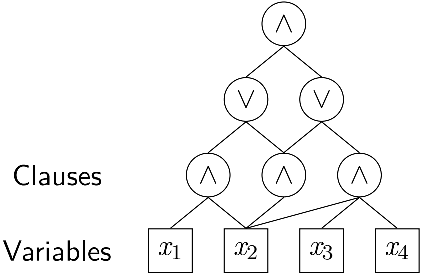

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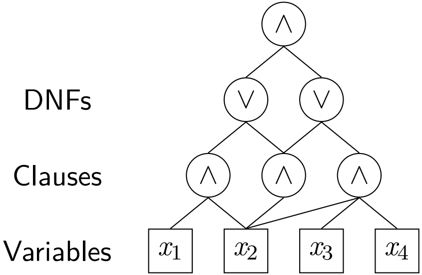

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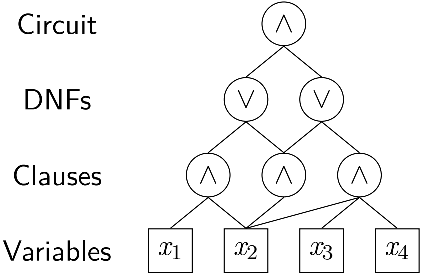

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- Call input bits “variables”.
- First level, AND gates “clauses”.
- Second level, OR gates “DNFs”.
- Third level, AND gate “circuits”.

Biased Coin Distributions

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Example: $D_{1/3}$ with 3 coins:

outputs	probabilities
111	$\left(\frac{1}{3}\right)^3$
$011, 101, 110$	$\left(\frac{1}{3}\right)^2 \frac{2}{3}$
$100, 010, 001$	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$
000	$\left(\frac{2}{3}\right)^3$

By central limit theorem, with probability almost one half, D_ϵ has less than ϵ fraction ones.

Restriction

Definition

We say $\rho \in \{0, 1, *\}^n$ is a restriction on n bits. We say the size of ρ, $|\rho|$, is the number of 1s and 0s in ρ.
If $f : \{0, 1\}^n \rightarrow \{0, 1\}$, then define $f \restriction_\rho$ as the function where the values from ρ are passed into f where it is 1 or 0, and otherwise the corresponding variable from the argument is passed in.

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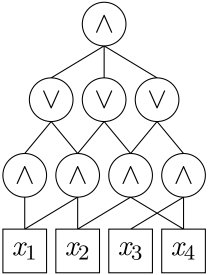

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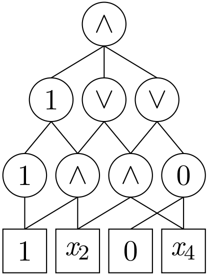

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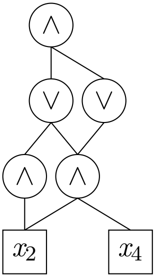

Monotone Lower Bound Idea

Idea: Lower bound the fan in at each layer.
Pretend $\epsilon \in (0, 1/2)$ is constant for simplicity. Let $\alpha = \frac{\epsilon}{\ln(1/\epsilon)}$.

1. From Viola [4], clauses have size $\frac{\ln(n)}{\ln(1/\epsilon)}$.

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2. DNFs have size $\tilde{\Omega}(n^{1+\alpha})$.
3. Circuit has $\tilde{\Omega}(n^{2+\alpha})$ clauses.

Greedy Set Cover

Theorem

Let $S = \{S_1, ..., S_m\}$ *be subsets of* $[n]$ *where each* $i \in [m]$ *has* $|S_i| \geq w$*. Then for any* $t \in [n]$ *there is some* $T \subseteq [n]$ *with* $|T| = t$ *so that* T *intersects all but at most*

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In particular, if

- S is the set of clauses in a monotone DNF, F, and
- ρ is some restriction restricting variables in T to 0,

then $|F\restriction_\rho| \leq |F|e^{-w\frac{t}{n}}$ variables remaining.

Monotone DNF Size

Theorem

Let $\epsilon \in (0, 1/2)$ *and monotone DNF* F *be such that*

- *For all x with less than* ϵn *zeros,* $F(x) = 1$*.*
- $\Pr[F(D_\epsilon) = 0] \geq$ *poly*$(1/n)$.

Then F has $n^{1+\alpha}$ *clauses for some* $\alpha = \Omega(\frac{\epsilon}{\ln(1/\epsilon)})$.

All DNFs in circuit must satisfy condition 1.

But For DNF to “help” by much, it must satisfy condition 2.

Monotone Circuit Size Lower Bounds

Theorem

Depth-3 Circuit C solving ϵ-promise majority has size $n^{2+\Omega\left(\frac{\epsilon}{\ln(1/\epsilon)}\right)}$.

Idea: Eliminate many DNFs with few clauses.

Can eliminate too many DNFs if there are not enough clauses.

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- Remove large clauses.
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Solution: Focus on large DNFs during elimination.

Insight: Some large DNF must survive if few variables fixed.

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- Circuit lower bounds, works!
  - At worst, might eliminate or shrink DNFs and clauses early.
  - But circuit still solves a promise problem, so it still has large DNFs after restriction.

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Use greedy set cover algorithm to choose variables like monotone case.

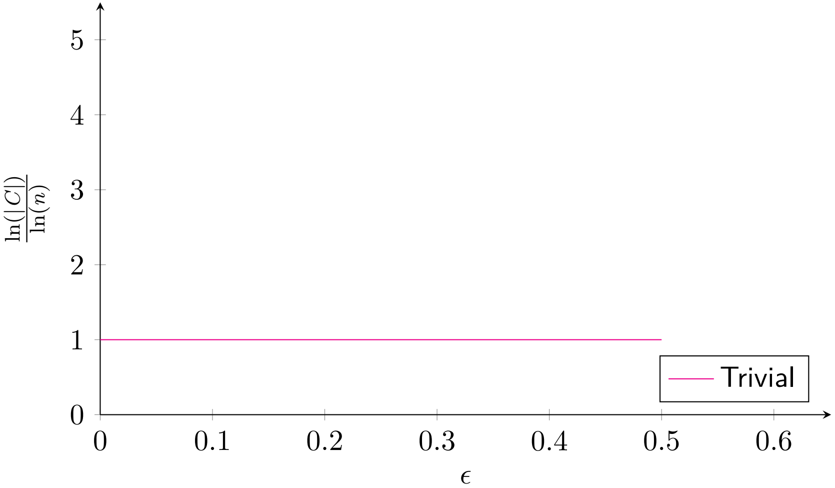
Instead of just setting them to 0, we set them to 1 with probability ϵ.

Then by Chernoff bounds, its likely that we eliminate many clauses.

By definition, restricting the rest of the variables is the same as using D_ϵ.

Depth-3 Bounds, Constant ϵ

Note: Graphs *slightly* adjusted for visibility.
Y-axis is c for circuit size n^c.



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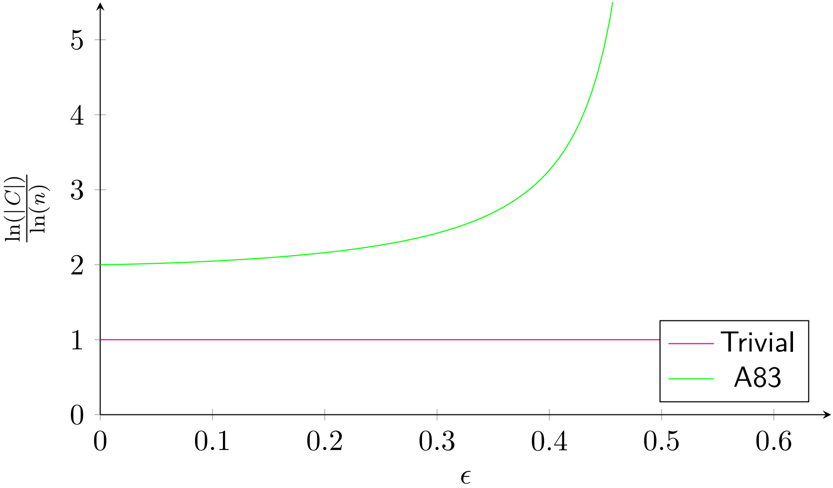

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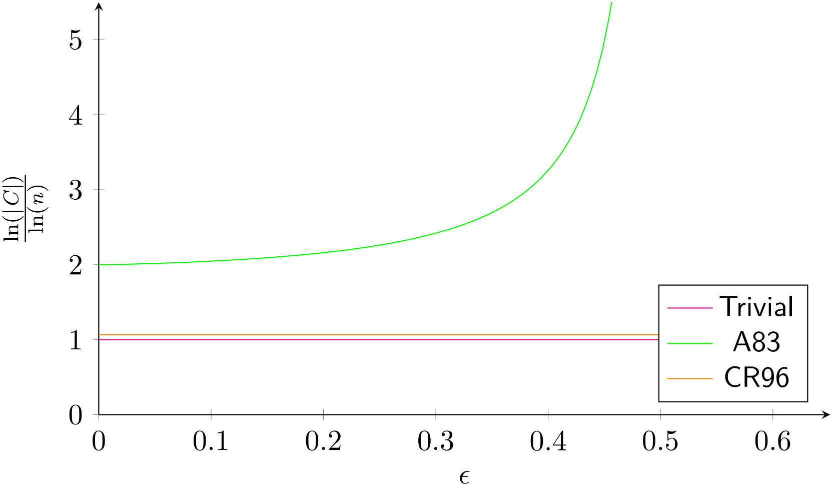

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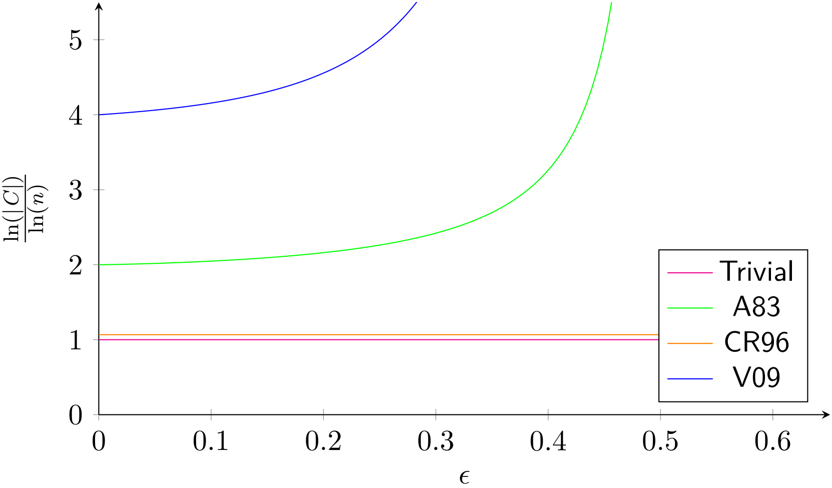

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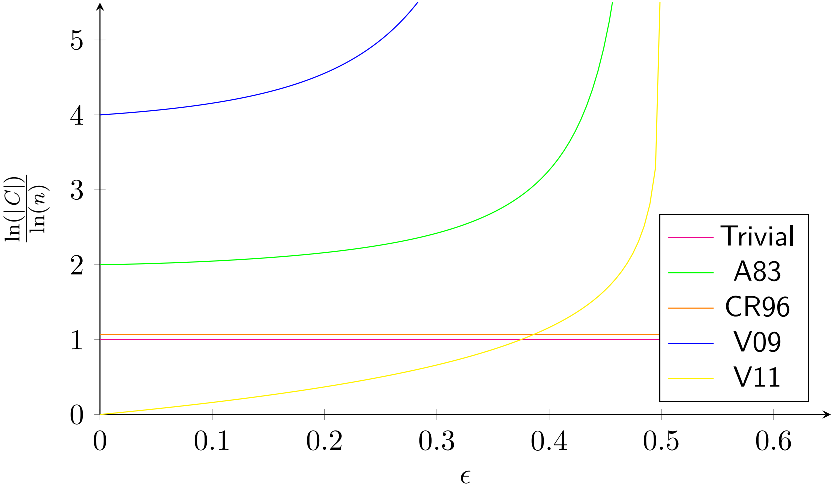

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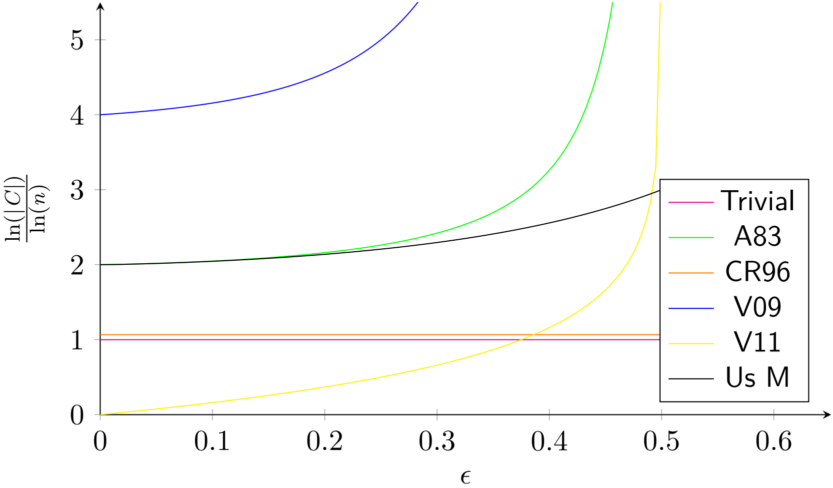

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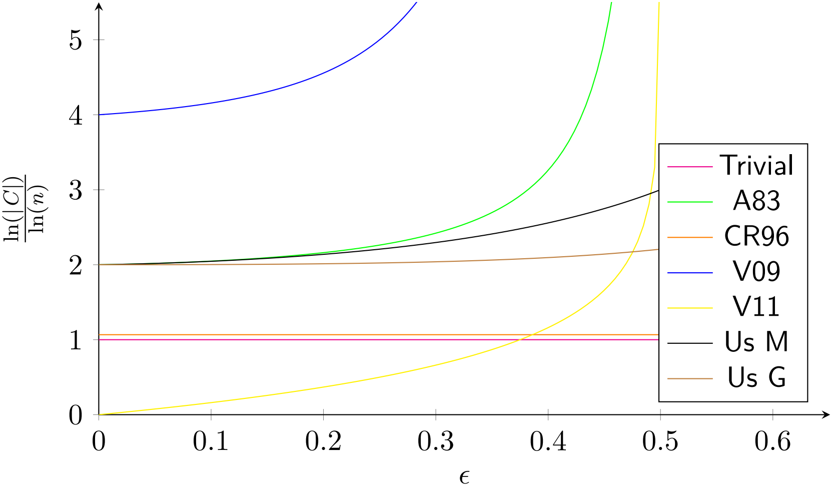

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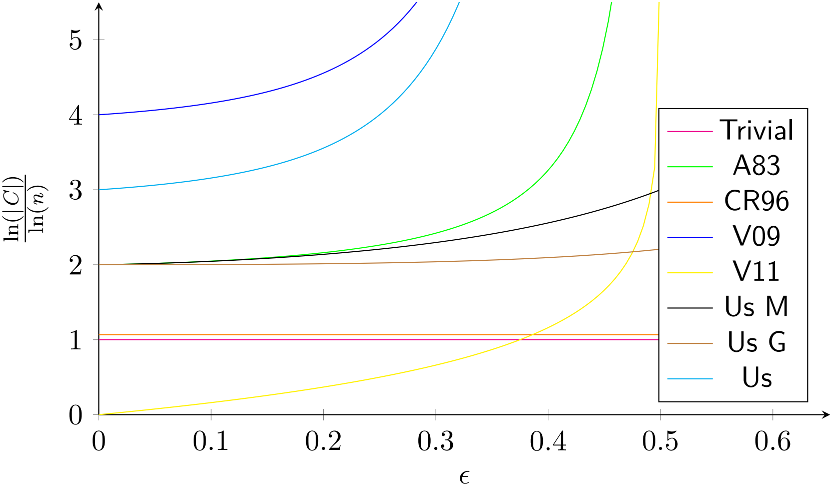

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