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Inductive Consequences in the Calculus of Constructions

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Conversion in CIC

CC + Inductive Definitions = Calculus of Inductive Constructions

conversion rule:

(conv) $$\frac{E \vdash e : t \qquad E \vdash t \stackrel{\star}{\longleftrightarrow} t'}{E \vdash e : t'}$$

$$\longrightarrow \; = \; \longrightarrow_\beta \cup \longrightarrow_\iota$$

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Larger conversion results in simpler and smaller proofs

Adding rewriting

`Symbol + : nat` $\longrightarrow$ `nat` $\longrightarrow$ `nat`

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```
Symbol + : nat → nat → nat
Rules 0 + m ⟶ m
      (S n) + m ⟶ S (n + m)
      n + 0 ⟶ n
      n + (S m) ⟶ S (n + m)
      (n + m) + l ⟶ n + (m + l).
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Goal (P m).
  exact fact.
Qed.
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$$\longrightarrow \;=\; \longrightarrow_\beta \cup \longrightarrow_R$$

Rewriting in the calculus of constructions

- rewriting packages in environments

$$E;\ \mathsf{Rew}(\, f\ :\ t;\ \{\ldots f\,\vec{l} \longrightarrow r\ldots\}\,);\ E'$$

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$$e[l\sigma]_p \longrightarrow_R e[r\sigma]_p$$

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Known metatheoretical properties of CC+Rew

- termination of $\longrightarrow_R \cup \longrightarrow_\beta$
  General Schema (F.Blanqui, PhD 2001),
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Complete definitions by rewriting

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Symbol + : nat → nat → nat
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```
Symbol K : ∀ (A:Set) (a:A) (P:eq A a a → Set),
           P (refl A a) → ∀ p: eq A a a, P p
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Complete subsystem and other rules

$$R' \begin{cases} R \begin{cases} f\ \vec{l_1} \longrightarrow r_1 \\ f\ \vec{l_2} \longrightarrow r_2 \end{cases} \\ f\ \vec{l_3} \longrightarrow r_3 \\ f\ \vec{l_4} \longrightarrow r_4 \\ f\ \vec{l_5} \longrightarrow r_5 \end{cases}$$

complete (often a pattern-matching definition of f)

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complete (often a pattern-matching definition of f)

Question

How are the rules in $R' \setminus R$ related to R ?

First order case

Other rules are inductive consequences:

Lemma

If $R \subseteq R'$ and they are both terminating, confluent and R is complete, then for all $l \longrightarrow r \in R' \setminus R$, for all σ closed, $l\sigma \stackrel{\star}{\longleftrightarrow}_R r\sigma$.

First order case

Other rules are inductive consequences:

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(conclusion from an old and simple theorem)

Our contribution

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Extension of inductive consequences lemma to higher-order rewriting on:

- non-functional inductive types
- functional inductive types

Inductive consequences for **non-functional** types

Inductive consequences for non-functional types

Theorem 1

If $R \subseteq R'$ and they are both terminating, confluent and R is complete and critical pairs are joinable without $\longrightarrow_{R' \setminus R}$ under a binder, then for all $l \longrightarrow r \in R' \setminus R$, for all σ closed, $l\sigma \stackrel{\star}{\longleftrightarrow}_R r\sigma$.

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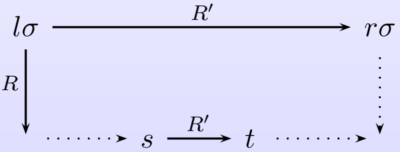

$$s[l'\sigma']_p \xrightarrow{R'} s[r'\sigma']_p$$

by induction on $\longrightarrow_{R'} \cup \longrightarrow_\beta \cup \rhd$

Inductive consequences for non-functional types

```
Inductive list : Set := nil : list | cons : bool→list→list
```

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`Inductive list : Set := nil : list | cons : bool→list→list`

`Symbol map : (bool→bool) → list → list`

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`Rules`

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Rules       map F nil ⟶ nil
            map F (cons a l) ⟶ cons (F a) (map F l)
```

Inductive consequences for non-functional types

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`Rules` map F nil $\longrightarrow$ nil

map F (cons a l) $\longrightarrow$ cons (F a) (map F l)

map $\lambda x.x$ l $\longrightarrow$ l

Inductive consequences for non-functional types

`Inductive list : Set := nil : list | cons : bool→list→list`

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Rules

$$R: \begin{array}{l} \texttt{map F nil} \longrightarrow \texttt{nil} \\ \texttt{map F (cons a l)} \longrightarrow \texttt{cons (F a) (map F l)} \\ \texttt{map } \lambda x.x \texttt{ l} \longrightarrow \texttt{l} \end{array}$$

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$$R : \begin{array}{l} \texttt{map F nil} \longrightarrow \texttt{nil} \\ \texttt{map F (cons a l)} \longrightarrow \texttt{cons (F a) (map F l)} \end{array}$$

$$R' \setminus R \ni \texttt{map } \lambda x.x \texttt{ l} \longrightarrow \texttt{l}$$

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`map` $\lambda x.x$ `nil`

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$\texttt{map } \lambda x.x \texttt{ (cons a l)}$

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$\downarrow_R$

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map $\lambda x.x$ (cons a l) $\longrightarrow_{R'}$ cons a l

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$R' \setminus R \ni$ `map` $\lambda x.x$ `l` $\longrightarrow$ `l`

`map` $\lambda x.x$ `nil` $\longrightarrow_{R'}$ `nil`

$\downarrow_R$

`nil`

`map` $\lambda x.x$ `(cons a l)` $\longrightarrow_{R'}$ `cons a l`

$\downarrow_R$

`cons (`$\lambda x.x$ `a) (map` $\lambda x.x$ `l)` $\longrightarrow_\beta$ `cons a (map` $\lambda x.x$ `l)`

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$$\texttt{map}\ \lambda x.x\ \texttt{nil} \longrightarrow_{R'} \texttt{nil}$$
$$\downarrow_R$$
$$\texttt{nil}$$

$$\texttt{map}\ \lambda x.x\ (\texttt{cons a l}) \longrightarrow_{R'} \texttt{cons a l}$$
$$\downarrow_R \qquad\qquad \uparrow^{R'}$$
$$\texttt{cons}\ (\lambda x.x\ \texttt{a})\ (\texttt{map}\ \lambda x.x\ \texttt{l}) \longrightarrow_{\beta} \texttt{cons a}\ (\texttt{map}\ \lambda x.x\ \texttt{l})$$

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Inductive consequences for non-functional types

Theorem 1

If $R \subseteq R'$ and they are both terminating, confluent and R is complete and critical pairs are joinable without $\longrightarrow_{R' \setminus R}$ under a binder,
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Symbol id : ord → ord
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      id (lim F) ⟶ lim (λ n. id (F n))
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      id (id x) ⟶ id x
```

Inductive consequences with functional inductive types

$$R \ni \texttt{id (lim F)} \longrightarrow \texttt{lim (}\lambda\texttt{ n. id (F n))}$$
$$R' \setminus R \ni \texttt{id (id x)} \longrightarrow \texttt{id x}$$

Inductive consequences with functional inductive types

$$R \ni \texttt{id (lim F)} \longrightarrow \texttt{lim } (\lambda \texttt{ n. id (F n))}$$
$$R' \setminus R \ni \texttt{id (id x)} \longrightarrow \texttt{id x}$$

$$\texttt{id (id (lim F))} \longrightarrow_{R'} \texttt{id (lim F)}$$
$$\downarrow_R$$
$$\texttt{id (lim } (\lambda \texttt{ n'. id (F n')))}$$

Inductive consequences with functional inductive types

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$$\downarrow_R$$

$$\texttt{id (lim (}\lambda\texttt{ n'. id (F n')))}$$

$$\downarrow_R$$

$$\texttt{lim (}\lambda\texttt{ n''. id ((}\lambda\texttt{ n'. id (F n')) n''))}$$

$$\downarrow_\beta$$

$$\texttt{lim (}\lambda\texttt{ n''. id (id (F n'')))} \longrightarrow_{R'} \texttt{lim (}\lambda\texttt{ n'.id (F n'))}$$

Inductive consequences with functional inductive types

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$\downarrow_R$

$\texttt{id (lim (}\lambda\ \texttt{n'. id (F n')))}$

$\downarrow_R$

$\texttt{lim (}\lambda\ \texttt{n''. id ((}\lambda\ \texttt{n'. id (F n')) n''))}$

$\downarrow_\beta$

$\texttt{lim (}\lambda\ \texttt{n''. id (id (F n'')))}$ $\longrightarrow_{R'}$ (to $\texttt{lim (}\lambda\ \texttt{n'.id (F n'))}$)

one needs one step of $\longrightarrow_{R'}$ under λ under `lim`.

Inductive consequences with functional inductive types

```
Symbol n2o : nat → ord

Rules n2o O ⟶ o
      n2o (S x) ⟶ s (n2o x)
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$l \longrightarrow r$

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`id (id x) ⟶ id x` $\quad l \longrightarrow r$

Now, for $\sigma = \{\mathtt{x} \mapsto \mathtt{lim}\ (\lambda\ \mathtt{n.}\ \mathtt{n2o}\ \mathtt{n})\}$ one has:

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`id (id x) ⟶ id x` $\qquad l \longrightarrow r$

Now, for $\sigma = \{$`x` $\mapsto$ `lim (λ n. n2o n)`$\}$ one has:

$l\sigma$`=id (id (lim (λ n. n2o n)))`
`⟶ id (lim (λ n'. id ((λ n. n2o n) n')))`
`⟶ id (lim (λ n'. id (n2o n')))`
`⟶ lim (λ n''. id ((λ n'. id (n2o n')) n''))`
`⟶ lim (λ n''. id (id (n2o n'')))`

$r\sigma$`=id (lim (λ n. n2o n))` $\qquad l\sigma \not\longleftrightarrow^{\star}_{R} r\sigma$
`⟶ lim (λ n'. id ((λ n. n2o n) n'))`
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Inductive consequences with functional inductive types

`Symbol n2o : nat` $\rightarrow$ `ord`

`Rules n2o O` $\longrightarrow$ `o`
`n2o (S x)` $\longrightarrow$ `s (n2o x)`

`id (id x)` $\longrightarrow$ `id x` $\qquad l \longrightarrow r$

Now, for $\sigma = \{$`x` $\mapsto$ `lim (`λ` n. n2o n)`$\}$ one has:

$l\sigma$=`id (id (lim (`λ` n. n2o n)))`
$\longrightarrow$ `id (lim (`λ` n'. id ((`λ` n. n2o n) n')))`
$\longrightarrow$ `id (lim (`λ` n'. id (n2o n')))`
$\longrightarrow$ `lim (`λ` n''. id ((`λ` n'. id (n2o n')) n''))`
$\longrightarrow$ `lim (`λ` n''. id (id (n2o n'')))`

$r\sigma$=`id (lim (`λ` n. n2o n))` $\qquad l\sigma \not\overset{\star}{\longleftrightarrow}_R r\sigma$
$\longrightarrow$ `lim (`λ` n'. id ((`λ` n. n2o n) n'))`
$\longrightarrow$ `lim (`λ` n'. id (n2o n'))`

Inductive consequences with functional inductive types

Theorem 2

If $R \subseteq R'$ and they are both terminating, confluent and R is complete and

- SN($\longrightarrow_{R'} \cup \longrightarrow_{\beta} \cup \rhd_c$)
- critical pairs are joinable with $\longrightarrow_{R' \setminus R}$ under constructors and $lambda$

then for all $l \longrightarrow r \in R' \setminus R$, for all σ closed, $l\sigma \sim_{\omega} r\sigma$.

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$\mathtt{S}\ \mathtt{x} \rhd_c \mathtt{x}$

$\mathtt{lim}\ \mathtt{F} \rhd_c \mathtt{F}\ t$ (for every typable t)

Inductive consequences with functional inductive types

Theorem 2

If $R \subseteq R'$ and they are both terminating, confluent and R is complete and

- R and R' are terminating by HORPO or General Schema
- critical pairs are joinable with $\longrightarrow_{R' \setminus R}$ under constructors and $lambda$

then for all $l \longrightarrow r \in R' \setminus R$, for all σ closed, $l\sigma \sim_\omega r\sigma$.

- ω-equivalence: $\sim_\omega$ is the least congruence containing $\overset{\star}{\longleftrightarrow}_R$ and ω-rule:

$$\frac{\forall \delta \text{ closed}, \quad s\delta \sim_\omega t\delta}{s \sim_\omega t}$$

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Conclusions

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Conclusions

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- Definitions by rewriting are made from a complete subset and “other” rules.
- We show that (under reasonable conditions) the “other” rules are inductive consequences of the complete subset.

General question:
What is the logical power of CC+Rew ?

Ideally: conservativity of CC+Rew over CIC+K+ext+?

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