The Optimal Fixed Point Combinator

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Interactive Theorem Proving (ITP'10) Edinburgh, 2010/07/14

Example: filter function for streams

Step 1: write a functional, e.g. for filter on streams

```
Definition Filter filter s :=
   let (x:::t) := s in
   if (P x) then (x ::: filter t) else (filter t).
```

// "filter" is a partial function mixing recursion and co-recursion

Step 2: construct its fixed point (non-constructively)

```
Definition filter := Fix Filter.  // return type inhabited
Definition filter := FixModulo (≈) Filter.  // actual
```

Step 3: prove a fixed point equation

```
Lemma filter_fix : forall s, infinitely_many P s ->
  filter s ≈ Filter filter s.
```

Step 4: used that equation to unfold the definition

```
  filter (x:::t)                        // rewrite filter_fix
≈ Filter filter (x:::t)                 // unfold Filter
≈ if (P x) then (x:::filter t) else (filter t)
```

Examples of recursive functions

Basic recursive function:

```
Definition Log log x :=
  if x <= 1 then 0 else 1 + log (x/2).

Definition log := FixModulo (=) Log.
Definition log := Fix Log. // equivalent to the line above
```

Nested recursion, e.g. the nested zero function:

```
Definition F f x =
   if x = 0 then 0 else f(f(x-1)).
```

// need to justify that f(x-1) is smaller than x

Higher-order recursion, e.g. a function modifying trees:

```
type tree = Leaf of nat | Node of list tree
Definition Incr incr x := match x with
  | Leaf n => Leaf (n+1)
  | Node xs => Node (List.map incr xs)
```

// need to justify that "incr" is applied to smaller trees

Examples of co-recursive values

Definition of co-recursive values:

```
Definition F s := 0 ::: map succ s.
Definition s := FixValModulo (≈) F. // 0:::1:::2:::3:::...
Lemma s_fix : s ≈ F s.
```

A trickier definition:

```
Definition F s := 2 ::: filter (≥ 1) s.
// F defines the stream "2:::2:::2:::...", because 2 ≥ 1.
```

An invalid definition:

```
Definition F s := 0 ::: filter (≥ 1) s.
// This functional does not admit a fixed point
Definition s := FixValModulo (≈) F.
// The stream s is unspecified
```

Program extraction is possible

The fixed point ombinators are not constructive.

They rely on Hilbert's epsilon operator, which does not have any computational equivalent.

Extraction towards a "let-rec" is possible:

```
Extract Constant Fix =>
  "(\bigf -> let x = bigf x in x)". // Haskell code
```

→ Partial correctness of the extracted code is to be expected (although I have not proved it formally)

→ Same trick used, e.g., by Bertot *et al* (2002)

Main fixed point approaches

– **Well-founded recursion:** for partial functions, the domain needs to appear explicitly.

– **Domain-predicate recursion** (Dubois & Donzeau-Gouge, Bove & Capretta) **and inductive graph predicate** (Krauss)**:** works for recursion but does not seem to extend to co-recursion.

– **Co-recursion with guard conditions:** definitions need to be modified so as to satisfy guard conditions either syntactic or type-based (e.g., work by Bertot and others), but such tricks are not always possible.

– **Contraction conditions:** allow proving the existence of a unique fixed point on a given domain, but does not help in constructing partial fixed point.

Ingredients and contribution

The combinator is built upon two ingredients:

1) Optimal fixed points

$\rightarrow$ First formalization of optimal fixed point theory

$\rightarrow$ First fixed point library using optimal fixed points

2) Contraction conditions

$\rightarrow$ Generalization of contr. conditions for co-recursion

$\rightarrow$ Unification of the various contraction conditions

Optimal fixed points

Consider the combinator for total recursive function:

```
Definition Fix F :=
  εf. (forall x, f x = F f x).
```

It generalizes to partial functions with something like:

```
Definition Fix D F :=
  εf. (forall x, D x -> f x = F f x).
```

However, the domain must be provided explicitly.

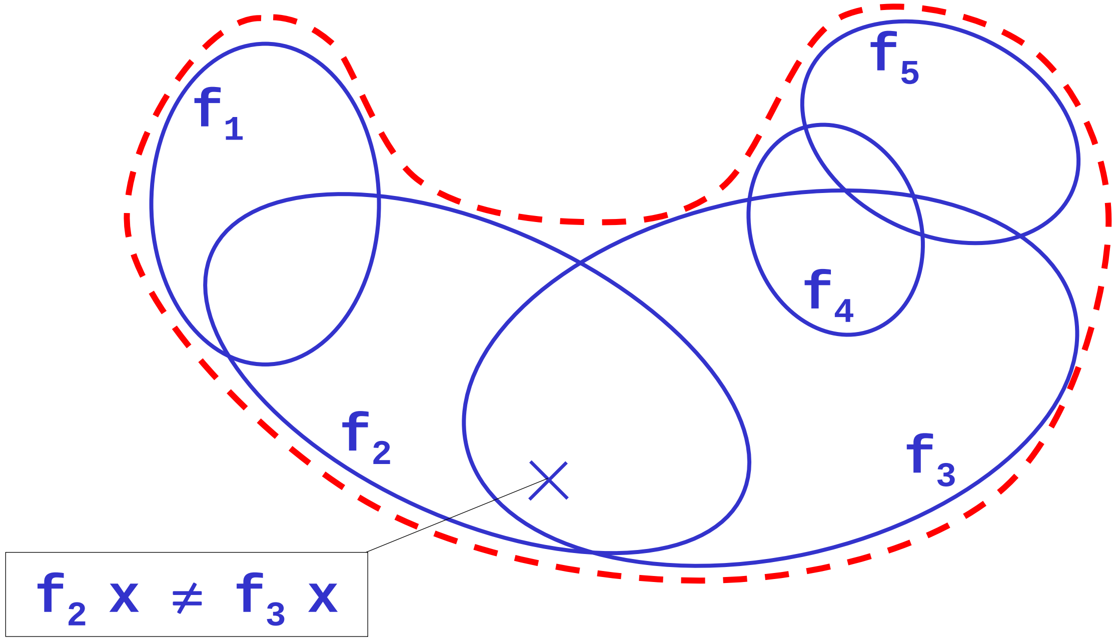
Question: is there a best possible domain D that can be deduced from the functional F alone?

Positive answer [Manna and Shamir, 1975]:

Any functional admits an *optimal* fixed point.

Domains of fixed points

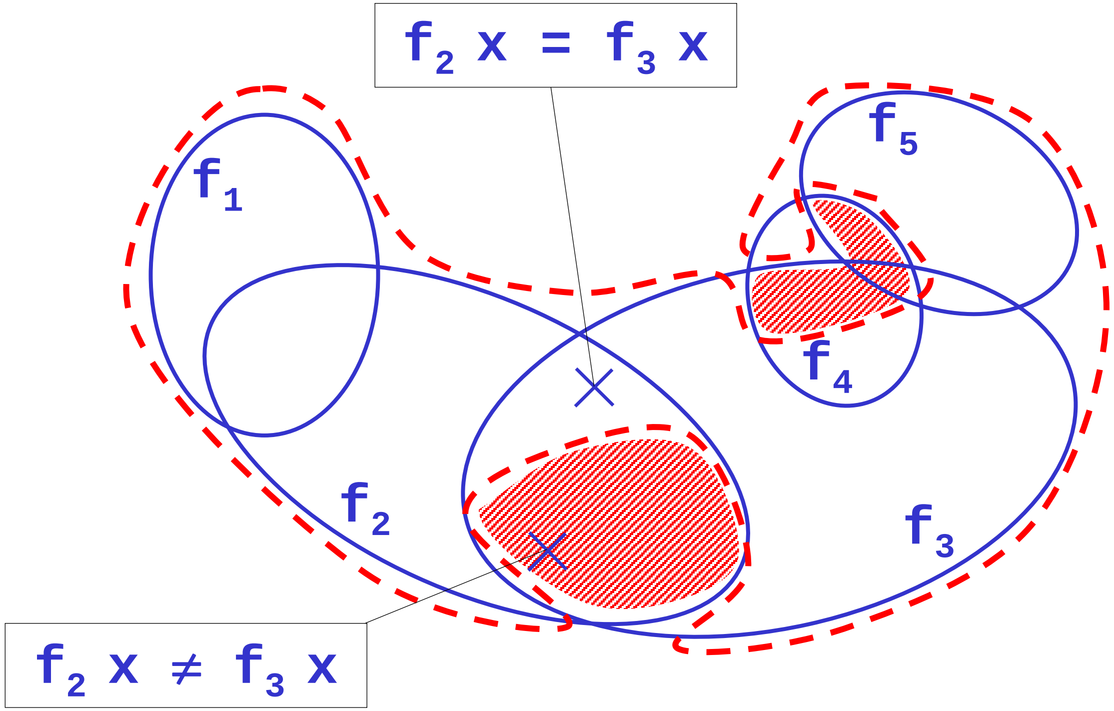
The union of the domains of all the fixed points might not be the domain of a fixed point:



$\rightarrow$ This generally happens with inconsistent fixed points

Domain of the optimal fixed point

The restriction to the set of arguments for which all fixed points return the same results:



$\rightarrow$ This domain admits exactly one fixed point, which captures the maximal amount of non-ambiguous information contained in the functional.

Optimal fixed point combinator

The optimal fixed point of a functional F is the largest generally-consistent fixed point of F.

(A fixed point of F is generally-consistent if it does not disagree with any other fixed point of F).

```
Definition Fix A B (F:(A->B)->(A->B)) : A->B :=
  εf. (optimal_fixed_point_of F f).
```

// Remark: the type B is required to be inhabited.

// Partial functions are represented in the logic as pairs of type (A→Prop)*(A->B). The optimal fixed point returned by the combinator Fix is undefined outside of the optimal domain.

Another construction (Gonthier, 2005)

```
Definition Fix A B F := fun x =>
  let f := εf.(∃D. fixed_point_on D F f ∧ x∈D) in (f x).
```

Contraction conditions

A contraction condition is a sufficient condition for a functional to admit a unique fixed point, expressing the fact that the functional *brings its arguments closer.*

– Guarantees unique fixed point in Banach spaces.

$$|| F(x) - F(y) || \leq \alpha \cdot || x - y || \qquad \text{with } \alpha < 1$$

– **Paulson** (1992): implement the theory of inductive definitions in Isabelle/HOL.

– **Matthews** (1999): formalize non-guarded co-recursive definitions.

– **Matthews & Krstić** (2003): formalize partial recursive functions with nested calls.

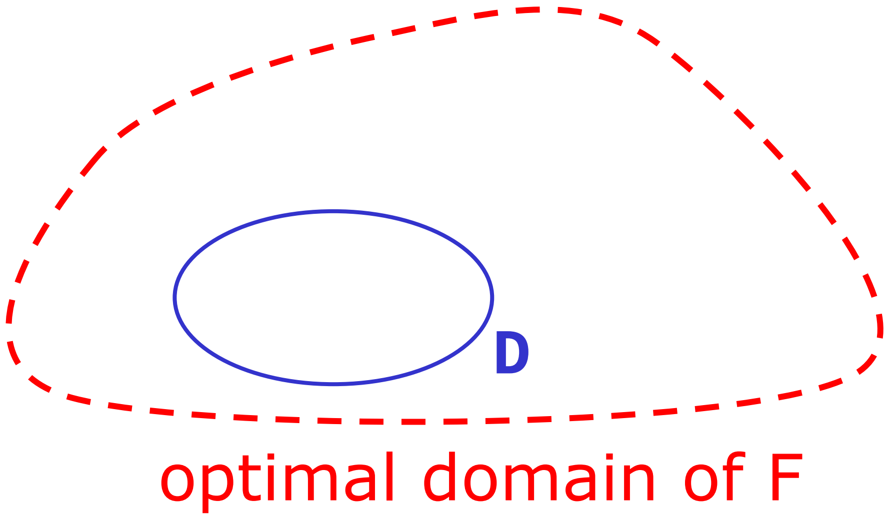
Fixed point theorems

How to use contraction conditions to reason on results of the optimal fixed point combinator:

1) Given a functional `F`, build `f := Fix F`.

2) Prove that `F` satisfies a contraction condition on some domain `D`.

3) Deduce that `f` satisfies the fixed point equation on `D`.




```
Theorem Fix_spec : forall F D f,
  f = Fix F -> contractive_on D F ->
  forall x, D x -> f x = F f x.
```

What's next

Application of the optimal fixed point combinator using existing contraction conditions:

- Total recursion
- Partial function
- Nested recursion
- Co-recursive values
- Co-recursive functions
- Mixed rec./co-recursive

(Supported but not presented: higher-order recursion)

Generalization and unification of the various contraction conditions:

- Generalization of the contraction condition
- Presentation of the unifying fixed point theorem

Treatment of total functions

Fixed point theorem for total recursive functions:

```
Lemma Fix_spec : forall f F R, well_founded R ->
  f = Fix F ->
  (forall f1 f2 x,
     (forall y, R y x -> f1 y = f2 y) ->
     F f1 x = F f2 x) ->
  (forall x, f x = F f x).
```

Illustration with the functional Log:

```
Hypothesis: forall y, y < x -> f1 y = f2 y

Goal: Log f1 x = Log f2 x

Goal:  (if x <= 1 then 0 else 1 + f1(x/2))
     = (if x <= 1 then 0 else 1 + f2(x/2))

Subgoal: x <= 1  |-  0 = 0
Subgoal: x > 1   |-  1 + f1(x/2) = 1 + f2(x/2)

Apply the hypothesis to y = x/2, and check (x/2) < x
```

Treatment of partial functions

Restriction to arguments from a domain D:

```
Lemma Fix_spec : forall f F R D, well_founded R ->
  f = Fix F ->
  (forall f1 f2 x, D x ->
     (forall y, D y -> R y x -> f1 y = f2 y) ->
     F f1 x = F f2 x) ->
  (forall x, D x -> f x = F f x).
```

→ The argument **x** is assumed to be in the domain **D**.

→ Recursive calls must be made to values **y** inside **D**.

→ The fixed point equation is available only on **D**.

Treatment of nested recursion

The basic contraction condition does not suffice. Consider for example the nested zero function:

```
Definition F f x =
  if x = 0 then 0 else f(f(x-1)).
```

→ For the outer recursive call `f(f(x-1))`, we need to know that the argument `f(x-1)` is smaller than `x`.

→ We need to know that the function `f` returns zero.

Adding an invariant [Matthews & Krstić, 2003]:

```
Lemma Fix_spec : forall f F R Q, well_founded R ->
  f = Fix F ->
  (forall f1 f2 x,
    (forall y, y < x -> f1 y = f2 y /\ Q y (f1 y)) ->
    F f1 x = F f2 x /\ Q x (F f1 x)) ->
  (forall x, f x = F f x /\ Q x (f x)).
```

Treatment of co-recursive values

Example:

```
Definition F s := 0 ::: map succ s. // 0:::1:::2:::3:::…
Definition s := FixValModulo (≈) F.
Lemma s_fix : s ≈ F s.
```

Fixed point combinator for values:

→ `FixValModulo (≈) F` picks a fixed point of `F` modulo `(≈)`

The insufficient, naive definition:

```
Definition FixValModulo (≈) F :=
  εx.(x ≈ F x).
```

The appropriate, standard definition:

```
Definition FixValModulo (≈) F :=
  εx.(forall y, y ≈ x -> y ≈ F y).
```

Contraction condition for streams

The contraction condition [Matthews, 1999]:

`forall i s1 s2, s1` $\approx_i$ `s2 -> F x1` $\approx_{i+1}$ `F s2`

implies the existence of a unique fixed point `s` modulo bisimilarity, where `(`$\approx_i$`)` relates two streams that are identical up to their i-th element.

Illustration with the stream of natural numbers:

`Hypothesis: s1` $\approx_i$ `s2`

`Goal: F s1` $\approx_{i+1}$ `F s2`

`Goal: 0 ::: map succ s1` $\approx_{i+1}$ `0 ::: map succ s2`

`Goal: map succ s1` $\approx_i$ `map succ s2`

`Exploit the fact that an application of map preserves the degree of similarity between two streams.`

General presentation of c.o.f.e.'s

Fixed point theorem from Matthews (1999) polished by di Gianantonio & Miculan (2003):

The contraction condition

```
forall i x1 x2,
  (forall j<i, x1 ≈j x2) ->
  F x1 ≈i F x2
```

ensures the existence of a unique fixed point **x** of **F** modulo **(≈)**, where:

- **F** has type **A→A**
- **I** is a type with a transitive well-founded relation **<**
- **≈** is the intersection of the equivalence relations $\approx_i$
- $(\approx_i)_{i:I}$ needs to be a *complete* family of relations

Treatment of co-recursive functions

The contraction condition for co-recursive functions given by Matthews (1999) leads to the following **fixed point theorem for co-recursive functions:**

```
Lemma FixModulo_spec : forall F f (≈_i)_{i∈I},
  f = FixModulo (≈) F -> cofe (≈_i)_{i∈I} ->
  (forall f1 f2 x i,
     (forall j<i, forall y, f1 y ≈_j f2 y) ->
     F f1 x ≈_i F f2 x) ->
  forall x, f x ≈ F f x.
```

Contraction condition for filter

Matthews (1999) also showed how to derive the **fixed point theorem for mixed rec/corec functions:**

```
Lemma FixModuloLexico_spec : forall (≈i)i∈I F f D,
  f = FixModulo (≈) F -> cofe (≈i)i∈I ->
  (forall f1 f2 x i, D x ->
    (forall y j, (j,y)<(i,x) -> D y -> f1 y ≈j f2 y) ->
    F f1 x ≈i F f2 x) ->
  forall x, D x -> f x ≈ F f x.
```

Illustration with the filter function:

→ `(j,y)<(i,x)` is a lexicographical comparison.

→ `i` decreases when the head value satisfies P.

→ `x` decreases when the next element satisfying P gets closer.

Co-recursion with an invariant

The tricky co-recursive definition:

```
Definition F s := 2 ::: filter (≥ 1) s.
```

New generalized form of contraction conditions:

forall x1 x2 i,
  $x1 \approx_i x2 \wedge Q\ i\ x1 \wedge Q\ i\ x2 \rightarrow$
  $F\ x1 \approx_{i+1} F\ x2 \wedge Q\ (i+1)\ (F\ x1)$

Illustration: it suffices to consider an invariant stating that the elements before index **i** are greater than **1**:

```
Definition Q i s := (∀j<i, nth j s ≥ 1).
```

Side-condition: the invariant Q has to be *continuous*.

Here, we need to show that if **Q i s** holds for any **i**, then **s** contains only values greater than **1**.

Key idea about invariants

Recursive definition → **specify results**

post-condition **Q x (f x)**

Co-recursive definition → **specify prefixes**

invariant **Q i s**

The unifying fixed point theorem

If the following hypotheses hold

- **F** is a functional of type **A->A** (where **A** is inhabited)
- **(A,I,<,$\approx_i$)** is a c.o.f.e.
- **Q** is a continuous property of type **I->A->Prop**
- The following contraction condition holds

$$\forall\ i\ x1\ x2,\ (\forall j<i,\ x1 \approx_j x2 \wedge Q\ j\ x1 \wedge Q\ j\ x2) \rightarrow F\ x1 \approx_i F\ x2 \wedge Q\ i\ (F\ x1)$$

Then we can deduce that

- **F** admits a unique fixed point **x** modulo $\approx$
- Moreover **x** satisfies the invariant: $\forall i,\ Q\ i\ x$

Several examples formalized

Recursion:	**Lines of proofs**
– log function	2
– gcd function	3
– div function	3
– nested zero function	3
– trees with lists of subtrees	4
– Ackermann's function	3
– McCarthy's function	8
Co-recursion: (≈ 100 lines to establish a new c.o.f.e.)	
– constant stream	3
– mutually-defined streams	9
– filter on streams	13
– "product" of infinite trees	24

Conclusion

1) Optimal fixed points:

– for long, a curiosity about circular *program* definitions

– the tool of choice to justify circular *logical* definitions

– allows to separate definitions from their justification

2) Contraction conditions:

– well-foundedness and productivity inside the logic

– support for a very large scope of circular definitions

– all contraction conditions derivable from a single one

(1) + (2) = Fix F

Thanks!

Extended version of the paper available from:

http://arthur.chargueraud.org/research/2010/fix