CHSH Inequality Experiments

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April 27, 2022

1 Introduction

Bell's Inequality in Quantum Mechanics disproves local hidden variable theories and addresses the EPR paradox. In this paper, we discuss briefly the background for this inequality and present our attempts at simulating and experimenting a version of Bell's Inequality called the Clauser-Horne-Shimony-Holt (CHSH) inequality. We conclude with the possible implications of our CHSH experiment and address loopholes.

2 Proof of Inequalities

In this section, we derive proofs for the classical and quantum bounds of the CHSH version of Bell's inequality. In particular we will show that the Bell Statistic S has maximum value of 2, where S is modeled by the following equation.

$S = |E(AB) + E(Ab)| + |E(aB) - E(ab)|$
where A, B, a, b are discrete random variables which can take the values $1, -1$

Assuming local realism, even allowing the presence of local hidden variables, then the quantum equivalent of this expression should also have an upper bound of 2. However, we will show that for certain choices of A, B, a, b we can exceed this bound and even reach a maximum value of $2\sqrt{2}$. In particular, we will be using the following quantum version of the inequality.

$S = \langle A \otimes B \rangle - \langle A \otimes b \rangle + \langle a \otimes B \rangle + \langle a \otimes b \rangle$
where $\langle X \otimes Y \rangle = \langle \psi | X \otimes Y | \psi \rangle$, which is the expectation value of $|\psi\rangle$ measured with $X \otimes Y$, also note that $|\psi\rangle = \dfrac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}$

To derive the classical bound we will first simplify the formula used to calculate the Bell Statistic (S).

$|E(AB) + E(Ab)| + |E(aB) - E(ab)|$
$= |A \cdot E(B + b)| + |a \cdot E(B - b)|$
$\leq E(|A(B + b)|) + E(|a(B - b)|)$
$= E(|A||B + b| + |a||B - b|)$
$= E(|B + b| + |B - b|)$

Next, we will find the value of $|B + b| + |B - b|$ given every possible combination of B and B' values.
$B = 1, b = 1$: $|1 + 1| + |1 - 1| = 2 + 0 = 2$
$B = 1, b = -1$: $|1 - 1| + |1 + 1| = 0 + 2 = 2$
$B = -1, b = 1$: $|-1 + 1| + |-1 - 1| = 0 + 2 = 2$
$B = -1, b = -1$: $|-1 - 1| + |-1 + 1| = 2 + 0 = 2$

Now, this expectation equals the following sum:
$2 \cdot Pr(B, b = 1, 1) + 2 \cdot Pr(B, b = 1, -1) + 2 \cdot Pr(B, b = -1, 1) + 2 \cdot Pr(B, b = -1, -1)$
$= 2 \cdot (Pr(B, b = 1, 1) + Pr(B, b = 1, -1) + Pr(B, b = -1, 1) + Pr(B, b = -1, -1))$
$= 2 \cdot 1 = 2$
The last step holds because these four probabilities cover the sample space and therefore their sum must be one.

Next, we will prove that the upper bound of 2 does not hold for a quantum system by counterexample. Let $A = Z$, $B = X$, a equal A rotated by $\pi/6$ radians, b equal B rotated by $\pi/6$ radians. Then, we calculate the following values:
$\langle A \otimes B \rangle = \sqrt{3}/2$
$\langle A \otimes b \rangle = -1/2$
$\langle a \otimes B \rangle = 1/2$
$\langle a \otimes b \rangle = \sqrt{3}/2$
Thus, $\langle A \otimes B \rangle - \langle A \otimes b \rangle + \langle a \otimes B \rangle + \langle a \otimes b \rangle = \sqrt{3}/2 - (-1/2) + 1/2 + \sqrt{3}/2 = \sqrt{3} + 1 \approx 2.732 > 2$

Now that we have shown the quantum system can exceed the classical bound, we will prove that the largest possible violation is $2\sqrt{2}$, which is known as Tsirelson's bound. Here, we will use the expression $AB + Ab + aB - ab$, which is equivalent to the one above with the values of the A and a variables swapped.

Proof of Tsirelson's upper bound ($2\sqrt{2}$)

Proof.

$$
\begin{aligned}
& AB + Ab + aB - ab \\
= \quad & \frac{1}{\sqrt{2}}\left(A^2 + a^2 + B^2 + b^2\right) && \text{(Lemma 0.1)} \\
& \left.\begin{aligned}
- \quad & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A-B) + (a-b)\Big)^2 \\
- \quad & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A-b) - (a+B)\Big)^2 \\
- \quad & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(a-B) + (A+b)\Big)^2 \\
- \quad & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(a+b) - (A+B)\Big)^2
\end{aligned}\right\} = \mathbf{a} && (\mathbf{a} \leq 0) \\
= \quad & \frac{1}{\sqrt{2}}\left(A^2 + a^2 + B^2 + b^2\right) + \mathbf{a} \\
\leq \quad & \frac{1}{\sqrt{2}}\left(A^2 + a^2 + B^2 + b^2\right) \\
= \quad & \frac{1}{\sqrt{2}}(1+1+1+1) && (A_i^2 = B_j^2 = \mathbb{I}, \text{ i.e., if the observables' outcomes are } \pm 1) \\
= \quad & 2\sqrt{2} \cdot \mathbb{I}
\end{aligned}
$$

□

Lemma 0.1.

$$
\begin{aligned}
& \frac{1}{\sqrt{2}}\left(A^2+a^2+B^2+b^2\right) \\
- & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A-B)+(a-b)\Big)^2 \\
- & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A-b)-(a+B)\Big)^2 \\
- & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(a-B)+(A+b)\Big)^2 \\
- & \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(a+b)-(A+B)\Big)^2 \\
\\
= & \frac{1}{\sqrt{2}}\left(A^2+a^2+B^2+b^2\right) \\
- & \frac{\sqrt{2}-1}{8}\Big((3+2\sqrt{2})\left(A^2-2AB+B^2\right)+a^2-2ab+b^2+(2\sqrt{2}+2)\left(Aa-aB-Ab+Bb\right)\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((3+2\sqrt{2})\left(A^2-2Ab+b^2\right)+a^2+2aB+B^2+(2\sqrt{2}+2)\left(Aa-ab+AB-Bb\right)\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((3+2\sqrt{2})\left(a^2-2aB+B^2\right)+A^2-2Ab+b^2+(2\sqrt{2}+2)\left(Aa-AB+ab-Bb\right)\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((3+2\sqrt{2})\left(a^2-2ab+b^2\right)+A^2-2AB+B^2-(2\sqrt{2}+2)\left(Aa+Ab+aB+Bb\right)\Big) \\
\\
= & \frac{1}{\sqrt{2}}\left(A^2+a^2+B^2+b^2\right) \\
- & \frac{\sqrt{2}-1}{8}\Big(2(4+2\sqrt{2})\left(A^2+a^2+B^2+b^2\right)\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((-8-8\sqrt{2})AB\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((-8-8\sqrt{2})aB\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((-8-8\sqrt{2})Ab\Big) \\
- & \frac{\sqrt{2}-1}{8}\Big((8+8\sqrt{2})ab\Big) \\
\\
= & \frac{1}{\sqrt{2}}\left(A^2+a^2+B^2+b^2\right)-\left(\frac{1}{\sqrt{2}}\left(A^2+a^2+B^2+b^2\right)-AB-aB-Ab+ab\right) \\
= & AB+Ab+aB-ab
\end{aligned}
$$

3 Experimental Setup, Experiments and Results

We next conduct three separate experiments to test the CHSH inequality.

We first perform a simulation using the *qasm_simulator* on IBM's Quantum Experience, modeled after the simulations run on *quito* (see sources). To prepare our experiment, we set A and a to the Z and X bases respectively, and perform a rotation on B and b to achieve a distinct pair of orthogonal bases. For our entangled wave function Ψ, we used the bell state $|\Phi^+\rangle$, and measured the four expectation values $\langle\Psi|A|\Psi\rangle$, for each basis.

We conducted our two other experiments independently of the IBM Quantum Experience. In the first, we simply tested we could replicate the results shown in the *qasm_simulator* by recreating the rotation matrix, bases, and psi state using numpy library, and using the built-in Kronecker product to tensor bases together. We got the same results as our noiseless simulation:

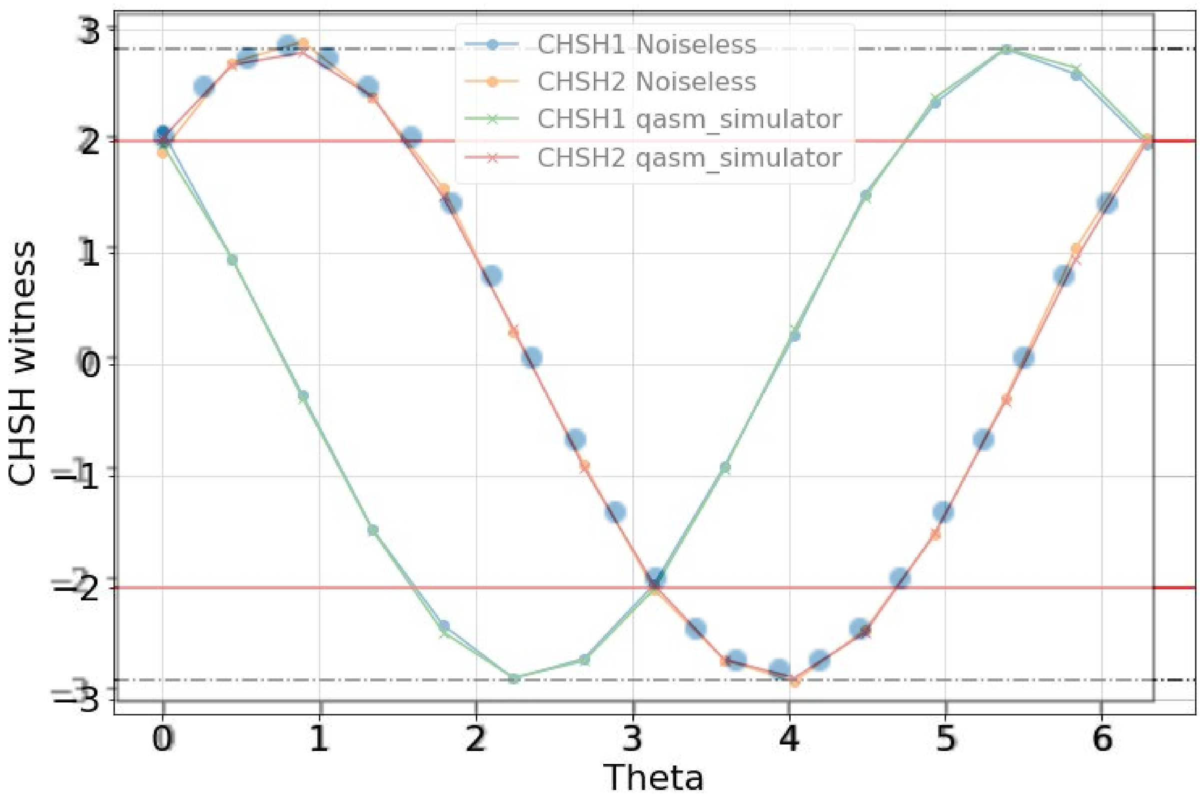

Figure 1: Overlayed Results from Simulation and Theoretical Experiment 1

In the second experiment, we generalized our bases to all possible unitaries, not only orthogonal. This allowed for us to see the general wave function rather than the extreme case. More results can be seen in the *Code_source* file.

4 Loopholes and Implications

Although we theoretically proved that the classical bound of expectation can be violated by quantum mechanics and even showed this on a real device (the *qasm_simulator*), there are a few loopholes that could potentially invalidate the conclusion of the CHSH inequality.

The first comes from the **detection loophole**, which relies on the inaccuracy of measurement devices letting hidden variables go undetected. This loophole can be addressed with high-fidelity machines, whose detectors have at least 83% accuracy.

The second loophole focuses on communication. Since it takes a small amount of time to measure any result, the **communication loophole** looks at the time it takes for one detector to somehow relay the result at lightspeed to the other detector, thereby affecting measurement results before any measurements are taken. In order to close this loophole, the two detectors should be separated by such a large distance and measured in such a small amount of time that light could not have traveled between them.

The third loophole relates to the concept of determinism in the universe. It states that even with two detectors set up very far apart, there is the possibility that communication between the detectors had happened in the past due to light striking both. Known as the **setting independence loophole**, a hypothesized solution would be to have both detectors use light from galaxies so far apart that light had not traveled between the two since the big bang. As a result, the detector settings will originate from sources not in contact from each other, ensuring independence for the detectors.

Overall, analyzing these loopholes can lead us to a broader discussion about the extent to which quantum mechanics can violate classical results by defying some of its key assumptions. For instance, the classical belief that the detectors cannot communicate faster than the speed of light is part of the idea that allows the original CHSH bound of 2 to hold. Yet given that we have been able to experimentally close this loophole and still achieve the same violation, we can conclude that some systems can only be accurately described by quantum mechanics. Thus, when the underlying assumptions of a mathematical statement like this inequality are based on classical ideas, the proof may be contradicted by quantum principles and hence real-world experiments.

5 Sources

Reproducing the experiment: https://qiskit.org/textbook/ch-demos/chsh.html
Context on local realism: http://www.quantumphysicslady.org/glossary/local-realism/
Hidden variables: http://www.quantumphysicslady.org/glossary/quantum-entanglement/
Implications of Bell's Theorem: https://brilliant.org/wiki/bells-theorem/
CHSH Inequality: https://laser.physics.sunysb.edu/_michaeldapolito/BellIneq/Bell_

Inequality_Write_up_revised2.pdf