DOMAIN RESTRICTION TRANSFORMATION

Alessandro Coglio

Kestrel Institute

2017

Restrict Domain of Non-Recursive Function

old function : $f(\bar{x}) \triangleq e(\bar{x})$ $\qquad \bar{x} = (x_1, \dots, x_n)$ $\qquad n > 0$

restricting predicate : $R \subseteq \mathcal{U}^n$

any value (irrelevant)

new function : $f'(\bar{x}) \triangleq \underline{\text{if}}\ R(\bar{x})\ \underline{\text{then}}\ e(\bar{x})\ \underline{\text{else}}\ \dots$

$\vdash \boxed{ff'}\ R(\bar{x}) \Rightarrow f(\bar{x}) = f'(\bar{x})$ – relation between f and f'

$R(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}) = e(\bar{x}) \underset{\delta_f}{=} f(\bar{x})$

QED

Restrict Domain of Recursive Function

old function: $f(\bar{x}) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ c(\bar{x}, f(\bar{d}(\bar{x})))$ $\quad \bar{x} = (x_1, \dots, x_n) \quad \bar{d}(\bar{x}) = (d_1(\bar{x}), \dots, d_n(\bar{x})) \quad n > 0$

$\boxed{T_f}\ \neg a(\bar{x}) \Rightarrow \mu_f(\bar{d}(\bar{x})) \prec_f \mu_f(\bar{x})$

restricting predicate : $R \subseteq \mathcal{U}^n$

condition: $\boxed{Rd}\ R(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow R(\bar{d}(\bar{x}))$ — preservation of R across recursive calls

new function: $f'(\bar{x}) \triangleq \underline{\text{if}}\ R(\bar{x})\ \underline{\text{then}}\ [\underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \text{else}\ c(\bar{x}, f'(\bar{d}(\bar{x})))]\ \underline{\text{else}}\ \dots$ any value (irrelevant)

$\mu_{f'}(\bar{x}) \triangleq \mu_f(\bar{x}) \qquad \prec_{f'} \triangleq \prec_f$

$\vdash \boxed{T_{f'}}\ R(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow \mu_{f'}(\bar{d}(\bar{x})) \prec_{f'} \mu_{f'}(\bar{x})$ — f' terminates

$$\neg a(\bar{x}) \xrightarrow[T_f]{} \begin{array}{ccc} \mu_f(\bar{d}(\bar{x})) & \prec_f & \mu_f(\bar{x}) \\ \delta_{\mu_{f'}} \| & \| \delta_{\prec_{f'}} & \| \delta_{\mu_{f'}} \\ \mu_{f'}(\bar{d}(\bar{x})) & \prec_{f'} & \mu_{f'}(\bar{x}) \end{array}$$

QED

$\vdash \boxed{ff'}\ R(\bar{x}) \Rightarrow f(\bar{x}) = f'(\bar{x})$ — relation between f and f'

induct f

base) $a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x})$

$a(\bar{x}),\ R(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}) = b(\bar{x})$

$\Rightarrow f(\bar{x}) = f'(\bar{x})$

step) $\neg a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}, f(\bar{d}(\bar{x})))$

$\neg a(\bar{x}),\ R(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}) = c(\bar{x}, f'(\bar{d}(\bar{x})))$

$\neg a(\bar{x}),\ R(\bar{x}) \xrightarrow{Rd} R(\bar{d}(\bar{x})) \xrightarrow[IH]{} f(\bar{d}(\bar{x})) = f'(\bar{d}(\bar{x}))$

$\Rightarrow f(\bar{x}) = f'(\bar{x})$

QED

Guards for the Non-Recursive Case

old function : $f(\bar{x}) \triangleq e(\bar{x})$

$\boxed{\surd f}$ $\gamma_{\gamma_f}(\bar{x}) \land \left[\gamma_f(\bar{x}) \Rightarrow \gamma_e(\bar{x}) \right]$

$\boxed{\surd R}$

condition : $\boxed{GR}$ $\gamma_f(\bar{x}) \Rightarrow \gamma_R(\bar{x})$ — R well-defined at least over the guard of f

new function : $f'(\bar{x}) \triangleq$ <u>if</u> $R(\bar{x})$ <u>then</u> $e(\bar{x})$ <u>else</u> ...

$\gamma_{f'}(\bar{x}) \triangleq \gamma_f(\bar{x}) \land R(\bar{x})$

$\vdash$ $\boxed{\surd f'}$

$$\omega_{f'}(\bar{x}) = \overset{\surd f}{\cancel{\gamma_{\gamma_f}(\bar{x})}} \land \overset{GR}{\cancel{\left[\gamma_f(\bar{x}) \Rightarrow \gamma_R(\bar{x})\right]}} \land \left[\gamma_f(\bar{x}) \land R(\bar{x}) \Rightarrow \cancel{\gamma_R(\bar{x})} \land \left[R(\bar{x}) \Rightarrow \cancel{\gamma_e(\bar{x})} \right] \land \left[\neg R(\bar{x}) \Rightarrow \cancel{\ldots} \right] \right]$$

GR

$\surd f$

QED

Guards for the Recursive Case

old function: $f(\bar{x}) \triangleq \text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } c(\bar{x}, f(\bar{d}(\bar{x})))$

$\gamma_{\bar{d}}(\bar{x}) = \gamma_{d_1}(\bar{x}) \wedge \dots \wedge \gamma_{d_n}(\bar{x})$

$\boxed{\sqrt{f}}$ $\quad \gamma_{\gamma f}(\bar{x}) \wedge [\gamma_f(\bar{x}) \Rightarrow \gamma_a(\bar{x}) \wedge [a(\bar{x}) \Rightarrow \gamma_b(\bar{x})] \wedge [\neg a(\bar{x}) \Rightarrow \gamma_{\bar{d}}(\bar{x}) \wedge \gamma_f(\bar{d}(\bar{x})) \wedge \gamma_c(\bar{x}, f(\bar{d}(\bar{x})))]]$

$\boxed{\sqrt{R}}$

condition: $\boxed{GR}$ $\gamma_f(\bar{x}) \Rightarrow \gamma_R(\bar{x})$ — as in the non-recursive case

new function: $f'(\bar{x}) \triangleq \text{if } R(\bar{x}) \text{ then } [\text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } c(\bar{x}, f'(\bar{d}(\bar{x})))] \text{ else } \dots$

$\gamma_{f'}(\bar{x}) \triangleq \gamma_f(\bar{x}) \wedge R(\bar{x})$ — as in the non-recursive case

$\vdash \boxed{\sqrt{f'}}$

$$\omega_{f'}(\bar{x}) = \overset{\sqrt{f}}{\cancel{\gamma_{\gamma f}(\bar{x})}} \wedge \overset{GR}{\cancel{[\gamma_f(\bar{x}) \Rightarrow \gamma_R(\bar{x})]}} \wedge$$

$$[\gamma_f(\bar{x}) \wedge R(\bar{x}) \Rightarrow \overset{GR}{\cancel{\gamma_R(\bar{x})}} \wedge$$

$$[R(\bar{x}) \Rightarrow \overset{\sqrt{f}}{\cancel{\gamma_a(\bar{x})}}] \wedge$$

$$\overset{\sqrt{f}}{\cancel{[a(\bar{x}) \Rightarrow \gamma_b(\bar{x})]}} \wedge$$

$$[\neg a(\bar{x}) \Rightarrow \overset{\sqrt{f}}{\cancel{\gamma_{\bar{d}}(\bar{x})}} \wedge \overset{\sqrt{f}}{\cancel{\gamma_f(\bar{d}(\bar{x}))}} \wedge \cancel{R(\bar{d}(\bar{x}))} \wedge \cancel{\gamma_c(\bar{x}, f'(\bar{d}(\bar{x})))}]] \wedge [\neg R(\bar{x}) \Rightarrow \cancel{\gamma \dots}]]$$

Rd

$\overset{ff'}{\Rightarrow} f(\bar{d}(\bar{x})) = f'(\bar{d}(\bar{x}))$

$\overset{\sqrt{f}}{\Rightarrow} \gamma_c(\bar{x}, f(\bar{d}(\bar{x}))) \longrightarrow \gamma_c(\bar{x}, f'(\bar{d}(\bar{x})))$

QED