

SCHEMATIC ALGORITHM TRANSFORMATION

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Generic Schema

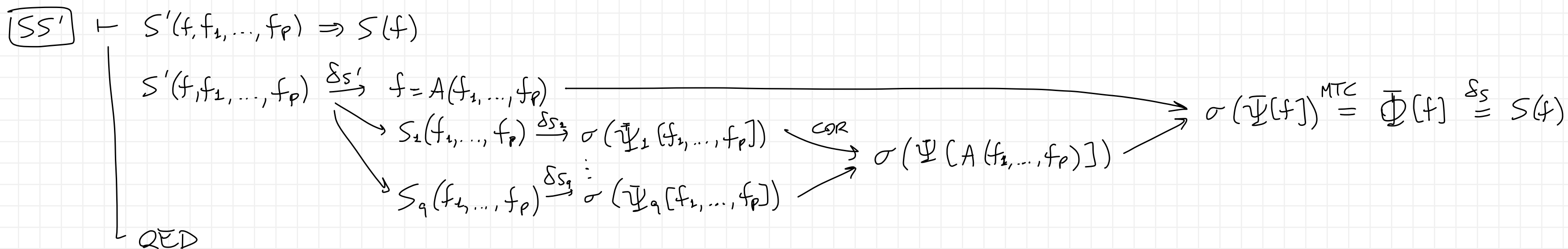
old specification: $S(f) \triangleq \Phi[f]$, $S \subseteq U^n \rightarrow U^m$

schematic algorithm $\left\{ \begin{array}{l} A(f_1, \dots, f_p) \triangleq \dots, \quad A \in (U^{n_1} \rightarrow U^{m_1}) \times \dots \times (U^{n_p} \rightarrow U^{m_p}) \rightarrow U^n \rightarrow U^m \quad \text{— algorithm function} \\ \boxed{\text{COR}} \vdash \underbrace{\Psi_1[f_1, \dots, f_p] \wedge \dots \wedge \Psi_q[f_1, \dots, f_p]}_{\text{each of these may actually depend on a strict subset of } \{f_1, \dots, f_p\}} \Rightarrow \Psi[A(f_1, \dots, f_p)] \quad \text{— correctness theorem} \end{array} \right\} \text{2nd-order}$

condition: $\boxed{\text{MTC}} \Phi[f]$ matches $\Psi[f]$, i.e. \exists substitution σ . $\Phi[f] = \sigma(\Psi[f])$

new specifications $\left\{ \begin{array}{l} S_1(f_1, \dots, f_p) \triangleq \sigma(\Psi_1[f_1, \dots, f_p]) \\ \vdots \\ S_q(f_1, \dots, f_p) \triangleq \sigma(\Psi_q[f_1, \dots, f_p]) \end{array} \right\}$ these may be easier to solve when they depend on strict subsets of $\{f_1, \dots, f_p\}$

$S'(f, f_1, \dots, f_p) \triangleq [f = A(f_1, \dots, f_p) \wedge S_1(f_1, \dots, f_p) \wedge \dots \wedge S_q(f_1, \dots, f_p)]$



$\hat{f}_1, \dots, \hat{f}_p$ solutions for $S_1, \dots, S_q \Rightarrow A(\hat{f}_1, \dots, \hat{f}_p)$ solution for S — final solution from sub-solutions

$\vdash S_1(\hat{f}_1, \dots, \hat{f}_p) \wedge \dots \wedge S_q(\hat{f}_1, \dots, \hat{f}_p) \quad \vdash S(A(\hat{f}_1, \dots, \hat{f}_p))$

see 'Specifications & Refinements' notes for background on S and its forms

Divide & Conquer List 0-1 Schema

$$A(g,h)(x, \bar{z}) \triangleq \text{if } \text{atom}(x) \text{ then } g(x, \bar{z}) \text{ else } h(\text{car}(x), \bar{z}, A(g,h)(\text{cdr}(x), \bar{z}))$$

$$z = z_1, \dots, z_p \quad p \geq 0$$

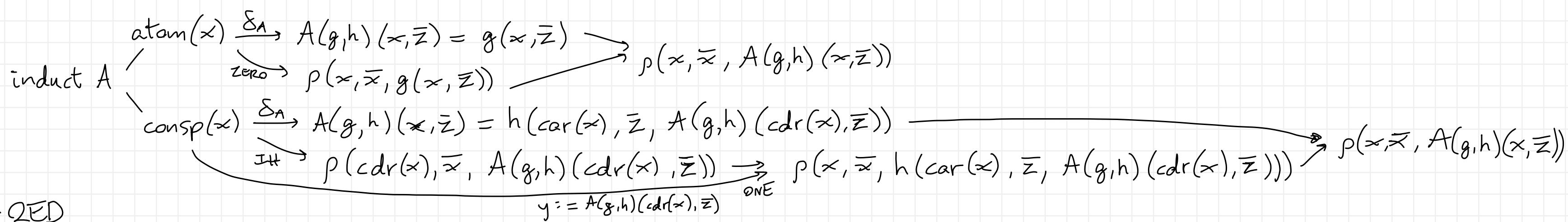
$$\mu_A(x, \bar{z}) \triangleq \text{len}(x) \quad \prec_A \triangleq \prec \quad \boxed{\tau_A} \vdash \neg \text{atom}(x) \Rightarrow \text{len}(\text{cdr}(x)) < \text{len}(x)$$

$$\boxed{\text{ZERO}} \quad \forall x, \bar{x}, \bar{z}. \text{atom}(x) \Rightarrow \rho(x, \bar{x}, g(x, \bar{z}))$$

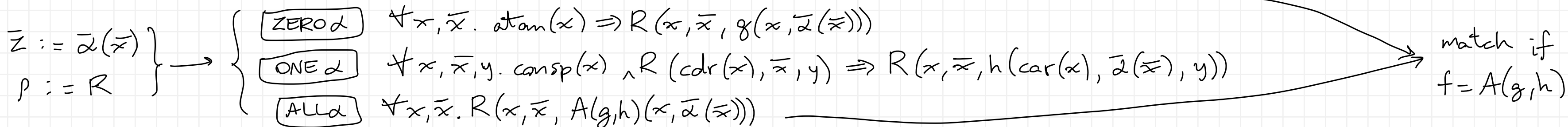
$$\boxed{\text{ONE}} \quad \forall x, \bar{x}, y, \bar{z}. \text{cons}(x) \wedge \rho(\text{cdr}(x), \bar{x}, y) \Rightarrow \rho(x, \bar{x}, h(\text{car}(x), \bar{z}, y))$$

$$\boxed{\text{ALL}} \quad \forall x, \bar{x}, \bar{z}. \rho(x, \bar{x}, A(g,h)(x, \bar{z}))$$

$$\boxed{\text{COR}} \vdash \boxed{\text{ZERO}} \wedge \boxed{\text{ONE}} \Rightarrow \boxed{\text{ALL}}$$



applicable to specification form $\boxed{Rf\alpha} \quad S(f) = [\forall x, \bar{x}. R(x, \bar{x}, f(x, \bar{\alpha}(\bar{x})))]$



Divide & Conquer List 0-1-2 Schema

$A(g_0, g_1, h)(x, \bar{z}) \triangleq$ if $\text{atom}(x)$ then $g_0(x, \bar{z})$
else if $\text{atom}(\text{cdr}(x))$ then $g_1(\text{car}(x), \text{cdr}(x), \bar{z})$
else $h(\text{car}(x), \bar{z}, A(g_0, g_1, h)(\text{cdr}(x), \bar{z}))$

$\bar{z} = z_1, \dots, z_p \quad p \geq 0$

$\mu_A(x, \bar{z}) \triangleq \text{len}(x) \quad <_A \triangleq < \quad \boxed{\tau_A} \vdash \neg \text{atom}(x) \wedge \neg \text{atom}(\text{cdr}(x)) \Rightarrow \text{len}(\text{cdr}(x)) < \text{len}(x)$

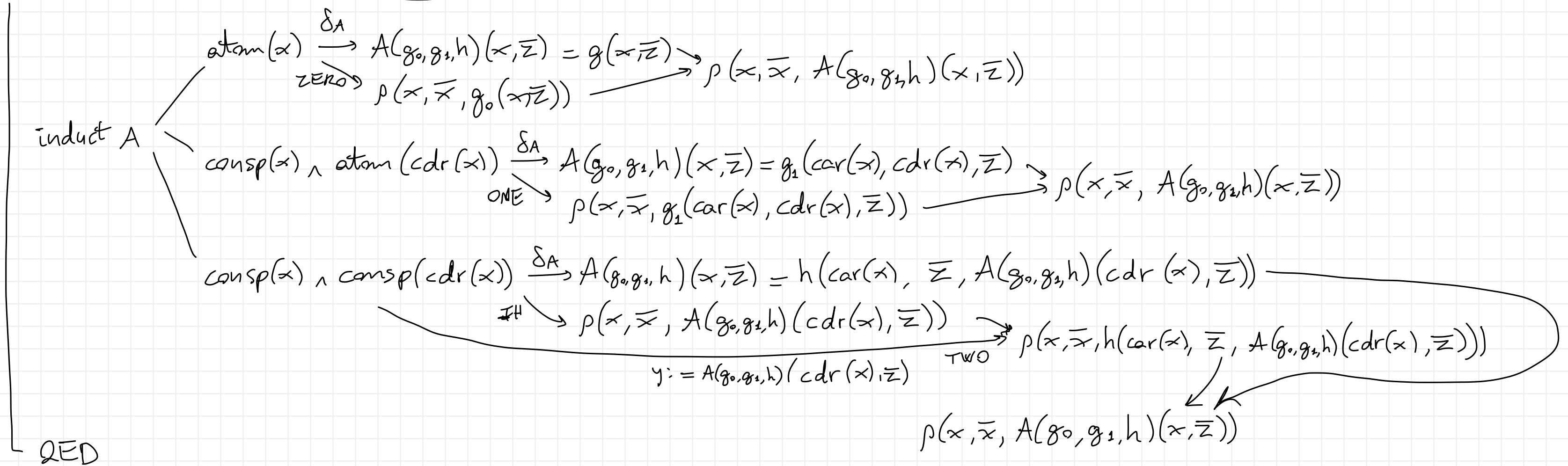
ZERO $\forall x, \bar{x}, \bar{z}. \text{atom}(x) \Rightarrow \rho(x, \bar{x}, g_0(x, \bar{z}))$

ONE $\forall x, \bar{x}, \bar{z}. \text{cons}(x) \wedge \text{atom}(\text{cdr}(x)) \Rightarrow \rho(x, \bar{x}, g_1(\text{car}(x), \text{cdr}(x), \bar{z}))$

TWO $\forall x, \bar{x}, y, \bar{z}. \text{cons}(x) \wedge \text{cons}(\text{cdr}(x)) \wedge \rho(\text{cdr}(x), \bar{x}, y) \Rightarrow \rho(x, \bar{x}, h(\text{car}(x), \bar{z}, y))$

ALL $\forall x, \bar{x}, \bar{z}. \rho(x, \bar{x}, A(g_0, g_1, h)(x, \bar{z}))$

COR $\vdash \boxed{\text{ZERO}} \wedge \boxed{\text{ONE}} \wedge \boxed{\text{TWO}} \Rightarrow \boxed{\text{ALL}}$



applicable to specification form **Rfd** — as in divide & conquer list 0-1 schema

Divide & Conquer Set 0-1 Schema

analogous to divide & conquer list 0-1 schema, with:

atom \longrightarrow empty

cons \longrightarrow 1 empty

car \longrightarrow head

cdr \longrightarrow tail

len \longrightarrow cardinality