SCHEMATIC ALGORITHM TRANSFORMATION

Alessandro Coglio

Kestrel Institute

Generic Schema

old specification : $S(f) \triangleq \Phi[f]$, $S \subseteq \mathcal{U}^n \to \mathcal{U}^m$

schematic algorithm $\begin{cases} A(f_1, \ldots, f_p) \triangleq \ldots \; , \quad A \in (\mathcal{U}^{n_1} \to \mathcal{U}^{m_1}) \times \ldots \times (\mathcal{U}^{n_p} \to \mathcal{U}^{m_p}) \to \mathcal{U}^n \to \mathcal{U}^m & \text{— algorithm function} \\ \boxed{\text{COR}} \vdash \underbrace{\Psi_1[f_1, \ldots, f_p] \wedge \cdots \wedge \Psi_q[f_1, \ldots, f_p]}_{\text{each of these may actually depend on a strict subset of } \{f_1, \ldots, f_p\}} \Rightarrow \Psi[A(f_1, \ldots, f_p)] & \text{— correctness theorem} \end{cases}$ $\Big\}$ 2^{nd}-order

condition : $\boxed{\text{MTC}}$ $\Phi[f]$ matches $\Psi[f]$, i.e. $\exists$ substitution σ . $\Phi[f] = \sigma(\Psi[f])$

new specifications $\begin{cases} \left.\begin{array}{l} S_1(f_1, \ldots, f_p) \triangleq \sigma(\Psi_1[f_1, \ldots, f_p]) \\ \vdots \\ S_q(f_1, \ldots, f_p) \triangleq \sigma(\Psi_q[f_1, \ldots, f_p]) \end{array}\right\} \text{these may be easier to solve when they depend on strict subsets of } \{f_1, \ldots, f_p\} \\ S'(f, f_1, \ldots, f_p) \triangleq [f = A(f_1, \ldots, f_p) \wedge S_1(f_1, \ldots, f_p) \wedge \cdots \wedge S_q(f_1, \ldots, f_p)] \end{cases}$

$\boxed{SS'}$ $\vdash S'(f, f_1, \ldots, f_p) \Rightarrow S(f)$

$$\begin{tikzcd}
S'(f, f_1, \ldots, f_p) \arrow[r, "\delta_{S'}"] \arrow[dr] \arrow[ddr] & f = A(f_1, \ldots, f_p) \arrow[rrr] & & & \sigma(\Psi[f]) \overset{\text{MTC}}{=} \Phi[f] \overset{\delta_S}{=} S(f) \\
& S_1(f_1, \ldots, f_p) \arrow[r, "\delta_{S_1}"] & \sigma(\Psi_1[f_1, \ldots, f_p]) \arrow[r, "\text{COR}"] & \sigma(\Psi(A(f_1, \ldots, f_p))) \arrow[ur] & \\
& S_q(f_1, \ldots, f_p) \arrow[r, "\delta_{S_q}"] & \sigma(\Psi_q[f_1, \ldots, f_p]) \arrow[ur] & &
\end{tikzcd}$$

QED

$\underbrace{\hat{f}_1, \ldots, \hat{f}_p \text{ solutions for } S_1, \ldots, S_q}_{\vdash S_1(\hat{f}_1, \ldots, \hat{f}_p) \wedge \cdots \wedge S_q(\hat{f}_1, \ldots, \hat{f}_p)} \Longrightarrow \underbrace{A(\hat{f}_1, \ldots, \hat{f}_p) \text{ solution for } S}_{\vdash S(A(\hat{f}_1, \ldots, \hat{f}_p))}$ — final solution from sub-solutions

see 'Specifications & Refinements' notes for background on S and its forms

Divide & Conquer List 0 – 1 Schema

$A(g,h)(x,\bar{z}) \triangleq \underline{\text{if}}\ atom(x)\ \underline{\text{then}}\ g(x,\bar{z})\ \underline{\text{else}}\ h(car(x), \bar{z}, A(g,h)(cdr(x),\bar{z}))$ $\quad z = z_1, \ldots, z_p \quad p \geq 0$

$\mu_A(x,\bar{z}) \triangleq len(x) \qquad \prec_A \triangleq < \qquad \boxed{\tau_A} \vdash \neg atom(x) \Rightarrow len(cdr(x)) < len(x)$

$\boxed{\text{ZERO}}\ \forall x, \bar{x}, \bar{z}.\ atom(x) \Rightarrow \rho(x, \bar{x}, g(x,\bar{z}))$

$\boxed{\text{ONE}}\ \forall x, \bar{x}, y, \bar{z}.\ consp(x) \wedge \rho(cdr(x), \bar{x}, y) \Rightarrow \rho(x, \bar{x}, h(car(x), \bar{z}, y))$

$\boxed{\text{ALL}}\ \forall x, \bar{x}, \bar{z}.\ \rho(x, \bar{x}, A(g,h)(x,\bar{z}))$

$\boxed{\text{COR}} \vdash \boxed{\text{ZERO}} \wedge \boxed{\text{ONE}} \Rightarrow \boxed{\text{ALL}}$

induct A

- $atom(x)$
  - $\xrightarrow{\delta_A} A(g,h)(x,\bar{z}) = g(x,\bar{z})$
  - $\xrightarrow{\text{ZERO}} \rho(x,\bar{x},g(x,\bar{z}))$
  - $\rightarrow \rho(x, \bar{x}, A(g,h)(x,\bar{z}))$
- $consp(x)$
  - $\xrightarrow{\delta_A} A(g,h)(x,\bar{z}) = h(car(x), \bar{z}, A(g,h)(cdr(x),\bar{z}))$
  - $\xrightarrow{\text{IH}} \rho(cdr(x), \bar{x}, A(g,h)(cdr(x),\bar{z})) \xrightarrow[\text{ONE}]{} \rho(x, \bar{x}, h(car(x), \bar{z}, A(g,h)(cdr(x),\bar{z})))$
  - $y := A(g,h)(cdr(x),\bar{z})$
  - $\rightarrow \rho(x,\bar{x},A(g,h)(x,\bar{z}))$

QED

applicable to specification form $\boxed{Rf\alpha}$ $S(f) = [\forall x, \bar{x}.\ R(x,\bar{x},f(x,\bar{\alpha}(\bar{x})))]$

$\left.\begin{array}{l} \bar{z} := \bar{\alpha}(\bar{x}) \\ \rho := R \end{array}\right\} \rightarrow \left\{\begin{array}{l} \boxed{\text{ZERO}\alpha}\ \forall x, \bar{x}.\ atom(x) \Rightarrow R(x, \bar{x}, g(x, \bar{\alpha}(\bar{x}))) \\ \boxed{\text{ONE}\alpha}\ \forall x, \bar{x}, y.\ consp(x) \wedge R(cdr(x), \bar{x}, y) \Rightarrow R(x, \bar{x}, h(car(x), \bar{\alpha}(\bar{x}), y)) \\ \boxed{\text{ALL}\alpha}\ \forall x, \bar{x}.\ R(x, \bar{x}, A(g,h)(x, \bar{\alpha}(\bar{x}))) \end{array}\right.$

match if $f = A(g,h)$

Divide & Conquer List 0-1-2 Schema

$A(g_0, g_1, h)(x, \bar{z}) \triangleq$ <u>if</u> $atom(x)$ <u>then</u> $g_0(x, \bar{z})$
<u>else</u> <u>if</u> $atom(cdr(x))$ <u>then</u> $g_1(car(x), cdr(x), \bar{z})$ $\qquad \bar{z} = z_1, \dots, z_p \qquad p \geq 0$
<u>else</u> $h(car(x), \bar{z}, A(g_0, g_1, h)(cdr(x), \bar{z}))$

$\mu_A(x, \bar{z}) \triangleq len(x) \qquad <_A \triangleq < \qquad \boxed{\tau_A} \vdash \neg atom(x) \wedge \neg atom(cdr(x)) \Rightarrow len(cdr(x)) < len(x)$

$\boxed{ZERO}$ $\forall x, \bar{x}, \bar{z}.\ atom(x) \Rightarrow \rho(x, \bar{x}, g_0(x, \bar{z}))$

$\boxed{ONE}$ $\forall x, \bar{x}, \bar{z}.\ consp(x) \wedge atom(cdr(x)) \Rightarrow \rho(x, \bar{x}, g_1(car(x), cdr(x), \bar{z}))$

$\boxed{TWO}$ $\forall x, \bar{x}, y, \bar{z}.\ consp(x) \wedge consp(cdr(x)) \wedge \rho(cdr(x), \bar{x}, y) \Rightarrow \rho(x, \bar{x}, h(car(x), \bar{z}, y))$

$\boxed{ALL}$ $\forall x, \bar{x}, \bar{z}.\ \rho(x, \bar{x}, A(g_0, g_1, h)(x, \bar{z}))$

$\boxed{COR} \vdash \boxed{ZERO} \wedge \boxed{ONE} \wedge \boxed{TWO} \Rightarrow \boxed{ALL}$

induct A

- $atom(x) \xrightarrow{\delta_A} A(g_0, g_1, h)(x, \bar{z}) = g(x, \bar{z}) \to \rho(x, \bar{x}, A(g_0, g_1, h)(x, \bar{z}))$
  $\xrightarrow{ZERO} \rho(x, \bar{x}, g_0(x, \bar{z})) \to$
- $consp(x) \wedge atom(cdr(x)) \xrightarrow{\delta_A} A(g_0, g_1, h)(x, \bar{z}) = g_1(car(x), cdr(x), \bar{z}) \to \rho(x, \bar{x}, A(g_0, g_1, h)(x, \bar{z}))$
  $\xrightarrow{ONE} \rho(x, \bar{x}, g_1(car(x), cdr(x), \bar{z})) \to$
- $consp(x) \wedge consp(cdr(x)) \xrightarrow{\delta_A} A(g_0, g_1, h)(x, \bar{z}) = h(car(x), \bar{z}, A(g_0, g_1, h)(cdr(x), \bar{z}))$
  $\xrightarrow{IH} \rho(x, \bar{x}, A(g_0, g_1, h)(cdr(x), \bar{z})) \to \rho(x, \bar{x}, h(car(x), \bar{z}, A(g_0, g_1, h)(cdr(x), \bar{z})))$
  $y := A(g_0, g_1, h)(cdr(x), \bar{z})$ $\quad$ TWO
  $\to \rho(x, \bar{x}, A(g_0, g_1, h)(x, \bar{z}))$

QED

applicable to specification form $\boxed{Rf\alpha}$ — as in divide & conquer list 0-1 schema

Divide & Conquer Set 0-1 Schema

analogous to divide & conquer list 0-1 schema, with:

- atom $\longrightarrow$ empty
- consp $\longrightarrow$ $\neg$ empty
- car $\longrightarrow$ head
- cdr $\longrightarrow$ tail
- len $\longrightarrow$ cardinality