TAIL RECURSION TRANSFORMATION

Alessandro Coglio

Kestrel Institute

Associative Binary Operator

some binary operator

old function: $f(\bar{x}) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ c(\bar{x}) * f(\bar{d}(\bar{x}))$ $\bar{x} = (x_1, \ldots, x_n)$ $\bar{d}(\bar{x}) = (d_1(\bar{x}), \ldots, d_n(\bar{x}))$ $n > 0$

$\boxed{\tau_f}$ $\neg a(\bar{x}) \Rightarrow \mu_f(\bar{d}(\bar{x})) \prec_f \mu_f(\bar{x})$

$$f(\bar{x}) = c(\bar{x}) * \overbrace{\left(c(\bar{d}(\bar{x})) * \left(c(\bar{d}^2(\bar{x})) * \ldots * \left(\overbrace{c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))}\right) \ldots \right) \right)}$$ $m \geq 0$ $\forall j \in \{0, \ldots, m-1\}.\ \neg a(\bar{d}^j(\bar{x}))$ $a(\bar{d}^m(\bar{x}))$

$\underbrace{\qquad}$ expansion of $f(\bar{x})$

condition: $\boxed{ASC}$ $u * (v * w) = (u * v) * w$ — associativity $\Rightarrow$ $(\mathcal{U}, *)$ is a semigroup

expansion of $f(\bar{x}) = \left(\ldots \left(\left(\overbrace{c(\bar{x}) * c(\bar{d}(\bar{x}))} \right) * c(\bar{d}^2(\bar{x})) \right) * \ldots * c(\bar{d}^{m-1}(\bar{x})) \right) * b(\bar{d}^m(\bar{x}))$

initially, if $\neg a(\bar{x})$,
set $r := c(\bar{x})$ and
update $\bar{x} := \bar{d}(\bar{x})$

then, while $\neg a(\bar{x})$,
update $r := r * c(\bar{x})$
and $\bar{x} := \bar{d}(\bar{x})$

finally, when $a(\bar{x})$,
return $r * b(\bar{x})$

if $a(\bar{x})$ initially,
return $b(\bar{x})$

calculate $f(\bar{x})$ when $m > 0$

calculate $f(\bar{x})$ when $m = 0$

new function: $f'(\bar{x}, r) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ r * b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), r * c(\bar{x}))$ — tail-recursive

$\mu_{f'}(\bar{x}, r) \triangleq \mu_f(\bar{x})$ $\prec_{f'} \triangleq \prec_f$

relation between f and f': $f(\bar{x}) = \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$ — proved next

$\vdash \boxed{\tau_{f'}}\quad \neg a(\bar{x}) \implies \mu_{f'}(\bar{d}(\bar{x}), r * c(\bar{x})) \prec_{f'} \mu_{f'}(\bar{x}, r) \qquad — f' \text{ terminates}$

$$\neg a(\bar{x}) \xrightarrow{\tau_f} \mu_f(\bar{d}(\bar{x})) \prec_f \mu_f(\bar{x})$$
$$\delta_{\mu_{f'}} \| \qquad \delta_{\prec_{f'}}\| \qquad \qquad \|\delta_{\mu_{f'}}$$
$$\mu_{f'}(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \prec_{f'} \quad \mu_{f'}(\bar{x}, r)$$

$\llcorner$ QED

$\vdash \boxed{f'f}\quad f'(\bar{x}, r) = r * f(\bar{x}) \qquad — \text{relation between } f' \text{ and } f \quad (f' \text{ in terms of } f)$

induct f'

base) $a(\bar{x}) \nearrow^{\delta_{f'}} f'(\bar{x}, r) = r * b(\bar{x}) = r * f(\bar{x})$
$\searrow^{\delta_f} f(\bar{x}) = b(\bar{x})$

step) $\neg a(x)$
$\searrow^{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
$\searrow^{\delta_{f'}} f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \overset{ASC}{=} r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x})$
IH $\searrow f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$

$\llcorner$ QED

$\vdash \boxed{ff'}\quad f(\bar{x}) = \underline{\text{if }} a(\bar{x}) \underline{\text{ then }} b(\bar{x}) \underline{\text{ else }} f'(\bar{d}(\bar{x}), c(\bar{x})) \qquad — \text{relation between } f \text{ and } f' \quad (f \text{ in terms of } f')$

$f(\bar{x}) \overset{\delta_f}{=} \underline{\text{if }} a(\bar{x}) \underline{\text{ then }} b(\bar{x}) \underline{\text{ else }} c(\bar{x}) * f(\bar{d}(\bar{x})) = \underline{\text{if }} a(\bar{x}) \underline{\text{ then }} b(\bar{x}) \underline{\text{ else }} f'(\bar{d}(\bar{x}), c(\bar{x}))$

$f'f \xrightarrow[\bar{x} := \bar{d}(\bar{x})]{r := c(\bar{x})} f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\llcorner$ QED

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wrapper : $\tilde{f}(\bar{x}) \triangleq \underline{\text{if }} a(\bar{x}) \underline{\text{ then }} b(\bar{x}) \underline{\text{ else }} f'(\bar{d}(\bar{x}), c(\bar{x})) \implies \vdash \boxed{f\tilde{f}}\quad f(\bar{x}) = \tilde{f}(\bar{x}) \implies \vdash \boxed{\tilde{f}f}\quad \tilde{f}(\bar{x}) = f(\bar{x})$

Associative Binary Operator with Left Identity

conditions $\begin{cases} \boxed{\text{ASC}} \quad u * (v * w) = (u * v) * w \quad \text{— associativity} \\ \boxed{\text{LI}} \quad b(\bar{x}) * u = u \quad \text{— left identity} \end{cases}$

$$f(\bar{x}) = c(\bar{x}) * \left(c(\bar{d}(\bar{x})) * \left(c(\bar{d}^2(\bar{x})) * \ldots * \left(c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x})) \right) \ldots \right) \right) \quad \text{— expansion of } f, \text{ as before}$$

$\|$

$$b(\bar{x}) * \left(c(\bar{x}) * \left(c(\bar{d}(\bar{x})) * \left(c(\bar{d}^2(\bar{x})) * \ldots * \left(c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x})) \right) \ldots \right) \right) \right)$$

ASC $\longrightarrow \|$

$$\left(\ldots \left(\left(\left(b(\bar{x}) * c(\bar{x}) \right) * c(\bar{d}(\bar{x})) \right) * c(\bar{d}^2(\bar{x})) \right) * \ldots * c(\bar{d}^{m-1}(\bar{x})) \right) * b(\bar{d}^m(\bar{x}))$$

start with $r := b(\bar{x})$
instead of $r := c(\bar{x})$

one more update $r := r * c(\bar{x})$ than before,
no need to update $\bar{x} := \bar{d}(\bar{x})$ initially

no initial split on $a(\bar{x})$,
because r starts as $b(\bar{x})$ instead of $c(\bar{x})$

new function: $f'(\bar{x}, r) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r * b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x}))$ — as in associative-only case

$\vdash \boxed{f'f} \quad f'(\bar{x}, r) = r * f(\bar{x})$ — as in associative-only case

$\vdash \boxed{ff'} \quad f(\bar{x}) = f'(\bar{x}, b(\bar{x}))$

$\quad f'f \xrightarrow{r := b(\bar{x})} f'(\bar{x}, b(\bar{x})) = b(\bar{x}) * f(\bar{x}) \overset{\text{LI}}{=} f(\bar{x})$

QED

wrapper: $\tilde{f}(\bar{x}) \triangleq f'(\bar{x}, b(\bar{x})) \implies \vdash \boxed{f\tilde{f}} \quad f(\bar{x}) = \tilde{f}(\bar{x})$

Associative Binary Operator with Right Identity

conditions $\begin{cases} \boxed{ASC} & u * (v * w) = (u * v) * w \quad\text{— associativity} \\ \boxed{RI} & u * b(\bar{x}) = u \quad\text{— right identity} \end{cases}$

$$f(\bar{x}) = c(\bar{x}) * \left(c(\bar{d}(\bar{x})) * \left(c(\bar{d}^2(\bar{x})) * \ldots * \left(\overline{c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))} \right) \ldots \right) \right) \quad\text{— expansion of } f, \text{ as before}$$

$ASC \longrightarrow \|$

$$\left(\ldots \left(\overline{\left(\overline{c(\bar{x}) * c(\bar{d}(\bar{x}))} \right) * c(\bar{d}^2(\bar{x}))} \right) * \ldots * c(\bar{d}^{m-1}(\bar{x})) \right) * b(\bar{d}^m(\bar{x})) \longleftarrow$$

if $m > 0$ $\|$

$$\left(\ldots \left(\overline{\left(\overline{c(\bar{x}) * c(\bar{d}(\bar{x}))} \right) * c(\bar{d}^2(\bar{x}))} \right) * \ldots * c(\bar{d}^{m-1}(\bar{x})) \right) \longleftarrow \begin{array}{l}\text{if } m>0, \\ \text{finally return } r \\ \text{instead of } r * b(\bar{x})\end{array}$$

if $m = 0$,
initially return $b(\bar{x})$,
as in associative-only case

$\Downarrow$

new function: $f'(\bar{x}, r) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ r\ \underline{else}\ f'(\bar{d}(\bar{x}), r * c(\bar{x}))$

$\mu_{f'}, \prec_{f'}, \tau_{f'}$ as before

$\vdash \boxed{f'f}\quad f'(\bar{x}, r) = r * f(\bar{x})$

induct f'

base) $a(\bar{x}) \begin{cases} \delta_{f'} \quad f'(\bar{x}, r) = r \overset{RI}{=} r * b(\bar{x}) = r * f(\bar{x}) \\ \delta_f \quad f(\bar{x}) = b(\bar{x}) \end{cases}$

step) $\neg a(\bar{x}) \begin{cases} \delta_f \quad f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x})) \\ \delta_{f'} \quad f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \overset{ASC}{=} r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x}) \\ \text{IH} \quad f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \end{cases}$

QED

$\vdash \boxed{ff'}\quad f(\bar{x}) = \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$ — as in associative-only case

wrapper: $\tilde{f}(\bar{x}) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), c(\bar{x})) \implies \vdash \boxed{f\tilde{f}}\quad f(\bar{x}) = \tilde{f}(\bar{x})$ — as in associative-only case

Associative Binary Operator with Left and Right Identity

conditions
$$\begin{cases} \boxed{\text{ASC}} & u * (v * w) = (u * v) * w \quad\quad - \text{ associativity} \\ \boxed{\text{LI}} & b(\bar{x}) * u = u \quad\quad\quad\quad\quad - \text{ left identity} \\ \boxed{\text{RI}} & u * b(\bar{x}) = u \quad\quad\quad\quad\quad - \text{ right identity} \end{cases} \Bigg\rangle \implies (\mathcal{U}, *, b_o) \text{ is a monoid}$$

$\vdash \boxed{\text{b. const}} \quad b(\bar{x}) = b(\bar{y})$

$\left. \begin{array}{l} b(\bar{x}) \underset{\text{RI}}{=} b(\bar{x}) * b(\bar{y}) \underset{\text{LI}}{=} b(\bar{y}) \\ \text{QED} \end{array} \right) \implies b_o \triangleq b(\bar{x}) \quad - \text{ constant value of } b$

$$f(\bar{x}) = c(\bar{x}) * \overbrace{\left(c(\bar{d}(\bar{x})) * \left(c(\bar{d}^2(\bar{x})) * \ldots * \overbrace{\left(c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x})) \right)} \ldots \right) \right)} \quad - \text{ expansion of } f, \text{ as before}$$

$\begin{array}{l} \text{ASC} \\ \text{LI} \\ \text{RI} \end{array} \Rrightarrow \|$

$$\ldots \left(\left(\overbrace{\left(\overbrace{(b(\bar{x}) * c(\bar{x}))} * c(\bar{d}(\bar{x})) \right)} * c(\bar{d}^2(\bar{x})) \right) * \ldots * c(\bar{d}^{m-1}(\bar{x})) \right.$$

start with $r := b(\bar{x})$ no initial $\bar{x} := \bar{d}(\bar{x})$ no final $r * b(\bar{x})$ no initial split on $a(\bar{x})$ $\Big\}$ "combine" LI and RI

     ↑ LI   ↑ LI   ↑ RI   ↑ LI

⇓

new function: $f'(\bar{x}, r) \triangleq \underline{\text{if }} a(\bar{x}) \underline{\text{ then }} r \underline{\text{ else }} f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad - \text{ as in associative-with-right-identity case}$

$\vdash \boxed{f'f} \quad f'(\bar{x}, r) = r * f(\bar{x}) \quad\quad\quad - \text{ as in associative-with-right-identity case}$

$\vdash \boxed{ff'} \quad f(\bar{x}) = f'(\bar{x}, b(\bar{x})) \quad\quad - \text{ as in associative-with-left-identity case}$

wrapper: $\tilde{f}(\bar{x}) \triangleq f(\bar{x}, b(\bar{x})) \implies \vdash \boxed{f\tilde{f}} \quad f(\bar{x}) = \tilde{f}(\bar{x}) \quad\quad - \text{ as in associative-with-left-identity case}$

Restriction of Operator Properties to a Domain

$D \subseteq \mathcal{U}$ — domain

conditions
$\begin{cases}
\boxed{Db} & a(\bar{x}) \Rightarrow D(b(\bar{x})) & \text{— domain closure for } b \\
\boxed{Dc} & \neg a(\bar{x}) \Rightarrow D(c(\bar{x})) & \text{— domain closure for } c \\
\boxed{D*} & D(u) \wedge D(v) \Rightarrow D(u*v) & \text{— domain closure for } * \\
\boxed{ASC} & D(u) \wedge D(v) \wedge D(w) \Rightarrow u*(v*w) = (u*v)*w & \text{— associativity} \\
\boxed{LI} & a(\bar{x}) \wedge D(u) \Rightarrow b(\bar{x}) * u = u & \text{— left identity (optional)} \\
\boxed{RI} & a(\bar{x}) \wedge D(u) \Rightarrow u * b(\bar{x}) = u & \text{— right identity (optional)}
\end{cases}$

$\vdash \boxed{Df} \quad D(f(\bar{x})) \qquad$ — domain closure for f

induct on f

$\quad$ base) $\quad a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x}) \searrow$
$\qquad\qquad\qquad \xrightarrow{Db} D(b(\bar{x})) \nearrow D(f(\bar{x}))$

$\quad$ step) $\quad \neg a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
$\qquad IH \begin{pmatrix} \xrightarrow{Dc} D(c(\bar{x})) \searrow \\ \searrow D(f(\bar{d}(\bar{x}))) \xrightarrow{D*} \end{pmatrix} D(c(\bar{x}) * f(\bar{d}(\bar{x}))) \nearrow D(f(\bar{x}))$

QED

$Db \wedge Dc \wedge D* \Rightarrow$ all the operands of $*$ in the expansion of $f(\bar{x})$ are in $D \Rightarrow$ ASC and RI apply

but LI applies only if $b(\bar{x})$ is such that $a(\bar{x})$ holds $\Rightarrow$ we cannot use $b(\bar{x})$ for any $\bar{x}$ in general

calculate, from any $\bar{x}$, some $b(\bar{x})$ such that $a(\bar{x})$ holds (i.e. go to the "bottom" of the recursion of f):

$$\beta(\bar{x}) \triangleq \text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } \beta(\bar{d}(\bar{x}))$$

$$\mu_\beta(\bar{x}) \triangleq \mu_f(\bar{x}) \qquad <_\beta \triangleq <_f$$

$\vdash \boxed{\tau_\beta} \quad \neg a(\bar{x}) \Rightarrow \mu_\beta(\bar{d}(\bar{x})) <_\beta \mu_\beta(\bar{x})$

$$\neg a(\bar{x}) \xrightarrow{\tau_f} \mu_f(\bar{d}(\bar{x})) <_f \mu_f(\bar{x})$$
$$\delta_{\mu_\beta} \| \qquad \delta_{<_\beta} \| \qquad \| \delta_{\mu_\beta}$$
$$\mu_\beta(\bar{d}(\bar{x})) <_\beta \mu_\beta(\bar{x})$$

$\llcorner$ QED

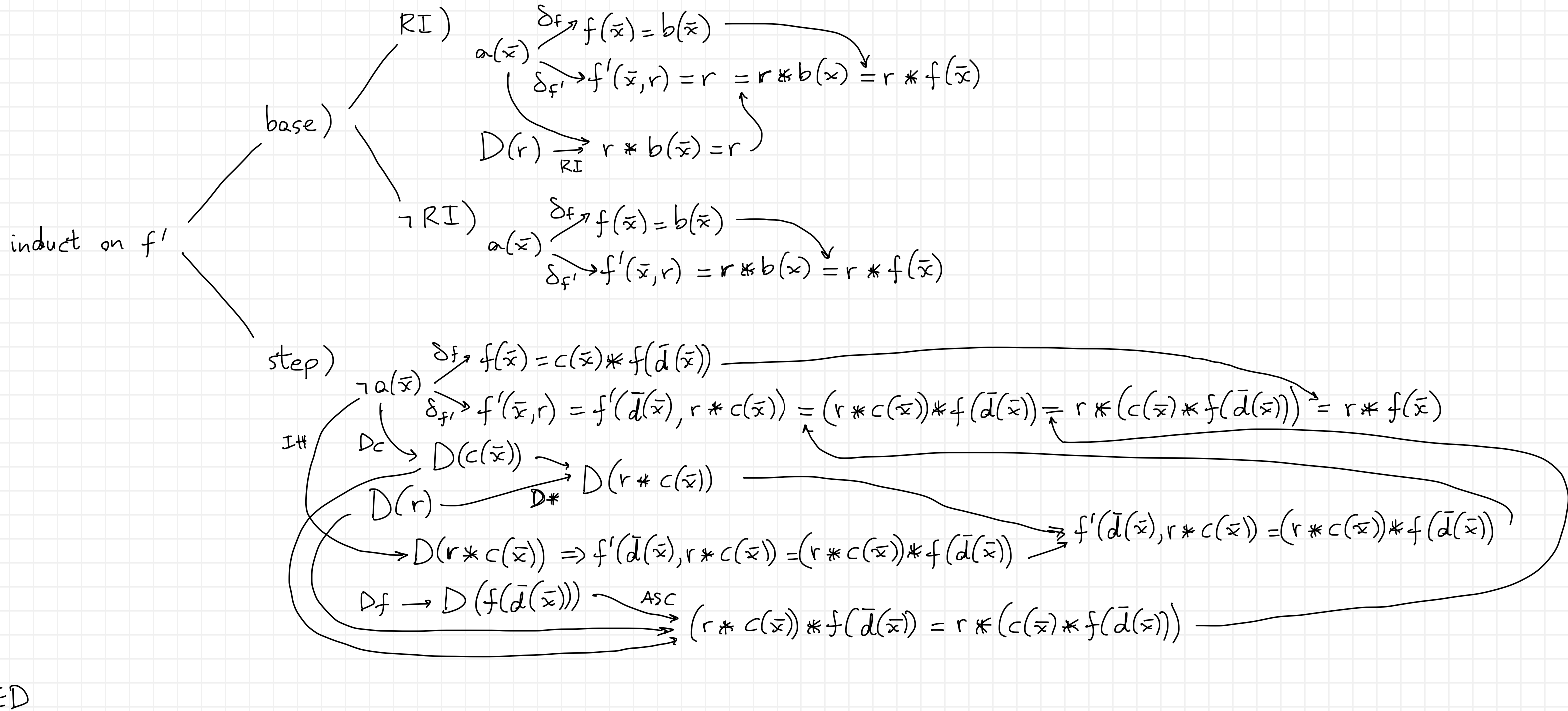$\vdash \boxed{D\beta} \quad D(\beta(\bar{x}))$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b(\bar{x})$
$\xrightarrow{Db} D(b(\bar{x})) \longrightarrow D(\beta(\bar{x}))$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$
$\xrightarrow{IH} D(\beta(\bar{d}(\bar{x}))) \longrightarrow D(\beta(\bar{x}))$

$\llcorner$ QED

LI $\Rightarrow$ $\vdash \boxed{LI\beta} \quad D(u) \Rightarrow \beta(\bar{x}) * u = u$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b(\bar{x})$
$D(u) \xrightarrow{LI} b(\bar{x}) * u = u$
$\longrightarrow \beta(\bar{x}) * u = u$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$
$D(u) \xrightarrow{IH} \beta(\bar{d}(\bar{x})) * u = u$
$\longrightarrow \beta(\bar{x}) * u = u$

$\llcorner$ QED

RI $\Rightarrow$ $\vdash \boxed{RI\beta} \quad D(u) \Rightarrow u * \beta(\bar{x}) = u$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b(\bar{x})$
$D(u) \xrightarrow{RI} u * b(\bar{x}) = u$
$\longrightarrow u * \beta(\bar{x}) = u$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$
$D(u) \xrightarrow{IH} u * \beta(\bar{d}(\bar{x})) = u$
$\longrightarrow u * \beta(\bar{x}) = u$

$\llcorner$ QED

new function: $f'(\bar{x}, r) \triangleq \begin{cases} \text{if } a(\bar{x}) \text{ then } r & \text{else } f'(\bar{d}(\bar{x}), r * c(\bar{x})) & \Leftarrow \text{RI} \\ \text{if } a(\bar{x}) \text{ then } r * b(\bar{x}) \text{ else } f'(\bar{d}(\bar{x}), r * c(\bar{x})) & \Leftarrow \neg \text{RI} \end{cases}$ — as when $D = \mathcal{U}$

$\vdash \boxed{f'f} \quad D(r) \Rightarrow f'(\bar{x}, r) = r * f(\bar{x})$

induct on f'

base)

RI) $a(\bar{x}) \begin{array}{l} \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x}) \\ \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = r = r * b(x) = r * f(\bar{x}) \end{array}$

$D(r) \xrightarrow[\text{RI}]{} r * b(\bar{x}) = r$

$\neg$ RI) $a(\bar{x}) \begin{array}{l} \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x}) \\ \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = r * b(x) = r * f(\bar{x}) \end{array}$

step) $\neg a(\bar{x}) \begin{array}{l} \xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x})) \\ \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) = r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x}) \end{array}$

IH

$D_c \searrow D(c(\bar{x})) \to D(r * c(\bar{x}))$

$D(r) \xrightarrow{\mathbf{D*}} D(r * c(\bar{x}))$

$D(r * c(\bar{x})) \Rightarrow f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \to f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$

$D_f \to D(f(\bar{d}(\bar{x}))) \xrightarrow{\text{ASC}} (r * c(\bar{x})) * f(\bar{d}(\bar{x})) = r * (c(\bar{x}) * f(\bar{d}(\bar{x})))$

QED

β and LI are not used in f' and $f'f$

$LI \implies$ $\vdash \boxed{f\,f'}$ $f(\bar{x}) = f'(\bar{x}, \beta(\bar{x}))$

$\qquad D\beta \longrightarrow D(\beta(\bar{x})) \searrow$

$\qquad f'f \xrightarrow{\quad r:=\beta(\bar{x}) \quad} f'(\bar{x}, \beta(\bar{x})) = \beta(\bar{x}) * f(\bar{x}) = f(\bar{x})$

$\qquad Df \longrightarrow D(f(\bar{x})) \underrightarrow{\qquad\qquad\qquad} \quad LI\beta$

$\qquad$ QED

$\qquad$ wrapper : $\tilde{f}(\bar{x}) \triangleq f'(\bar{x}, \beta(\bar{x})) \implies \vdash \boxed{f\,\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$ $\qquad$ — as when $D = \mathcal{U}$

$\neg LI \implies$ $\vdash \boxed{\bar{f}\,f'}$ $f(\bar{x}) = \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$

$\qquad f(\bar{x}) \stackrel{\delta_f}{=} \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ c(\bar{x}) * f(\bar{d}(\bar{x})) = \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$

$\qquad f'f \xrightarrow[\bar{x}:=\bar{d}(\bar{x})]{r:=c(\bar{x})} D(c(\bar{x})) \implies f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \neg a(\bar{x}) \implies f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\qquad Dc \longrightarrow \neg a(\bar{x}) \implies D(c(\bar{x})) \underrightarrow{\qquad\qquad\qquad\qquad}$

$\qquad$ QED

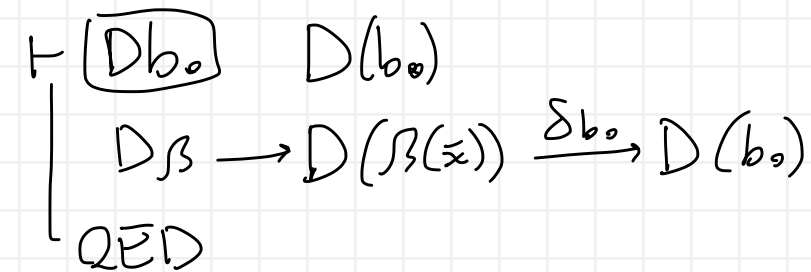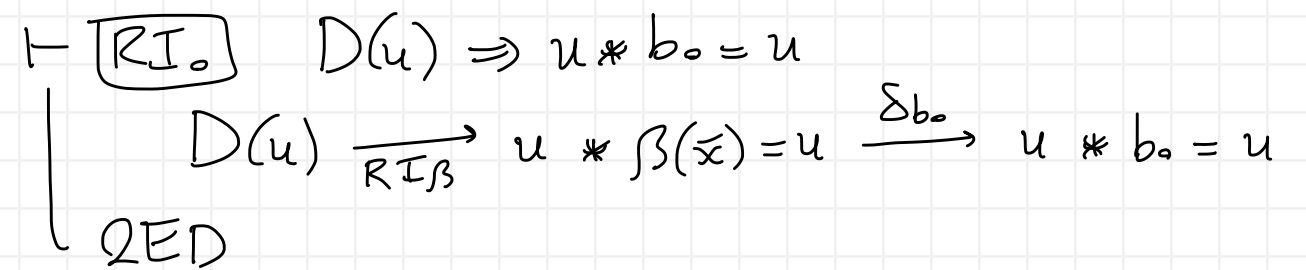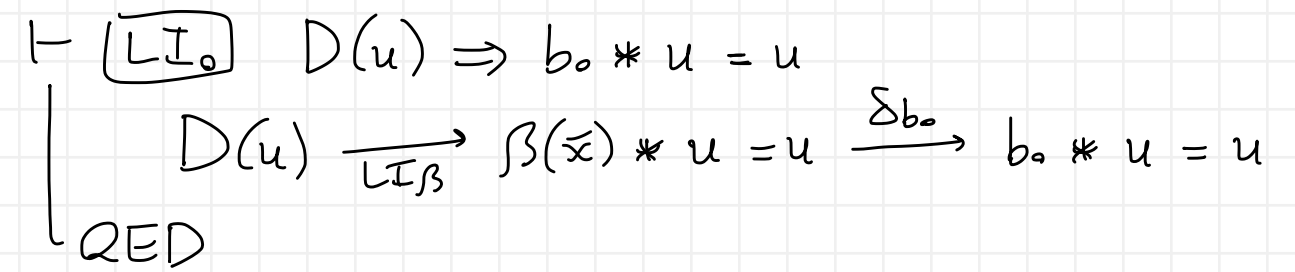$\qquad$ wrapper : $\tilde{f}(\bar{x}) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), c(\bar{x})) \implies \vdash \boxed{f\,\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$ $\qquad$ — as when $D = \mathcal{U}$

β is used only if LI holds ; only LIβ is used here, not RIβ

LI $\wedge$ RI $\implies$

$\vdash$ boxed[b.const] $a(\bar{x}) \wedge a(\bar{y}) \implies b(\bar{x}) = b(\bar{y})$

$a(\bar{x}) \xrightarrow{Db} D(b(\bar{x})) \xrightarrow{RI} b(\bar{x}) * b(\bar{y}) = b(\bar{x})$

$a(\bar{y}) \xrightarrow{Db} D(b(\bar{y})) \longrightarrow b(\bar{x}) * b(\bar{y}) = b(\bar{y})$ $\xrightarrow{} b(\bar{x}) = b(\bar{y})$

LI

$\llcorner$ QED

$\vdash$ boxed[βb] $a(\bar{y}) \implies \beta(\bar{x}) = b(\bar{y})$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta\beta} \beta(\bar{x}) = b(\bar{x})$

$a(\bar{y}) \xrightarrow[b.const]{} b(\bar{x}) = b(\bar{y})$ $\xrightarrow{} \beta(\bar{x}) = b(\bar{y})$

step) $\neg a(\bar{x}) \xrightarrow{\delta\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$

$a(\bar{y}) \xrightarrow{IH} \beta(\bar{d}(\bar{x})) = b(\bar{y})$ $\xrightarrow{} \beta(\bar{x}) = b(\bar{y})$

$\llcorner$ QED

$\vdash$ boxed[β.const] $\beta(\bar{x}) = \beta(\bar{y})$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta\beta} \beta(\bar{x}) = b(\bar{x})$

$\xrightarrow[\substack{\bar{x} := \bar{y} \\ \bar{y} := \bar{x}}]{\beta b} \beta(\bar{y}) = b(\bar{x})$ $\xrightarrow{} \beta(\bar{x}) = \beta(\bar{y})$

step) $\neg a(\bar{x}) \xrightarrow{\delta\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$

$\xrightarrow{IH} \beta(\bar{d}(\bar{x})) = \beta(\bar{y})$ $\xrightarrow{} \beta(\bar{x}) = \beta(\bar{y})$

$\llcorner$ QED

$b_0 \triangleq \beta(\bar{x})$ — constant value of β, i.e. of b under a

$\vdash$ boxed[LI_0] $D(u) \implies b_0 * u = u$

$D(u) \xrightarrow{LI\beta} \beta(\bar{x}) * u = u \xrightarrow{\delta b_0} b_0 * u = u$

$\llcorner$ QED

$\vdash$ boxed[RI_0] $D(u) \implies u * b_0 = u$

$D(u) \xrightarrow{RI\beta} u * \beta(\bar{x}) = u \xrightarrow{\delta b_0} u * b_0 = u$

$\llcorner$ QED

boxed[Db_0] $D(b_0)$

$D\beta \longrightarrow D(\beta(\bar{x})) \xrightarrow{\delta b_0} D(b_0)$

$\llcorner$ QED
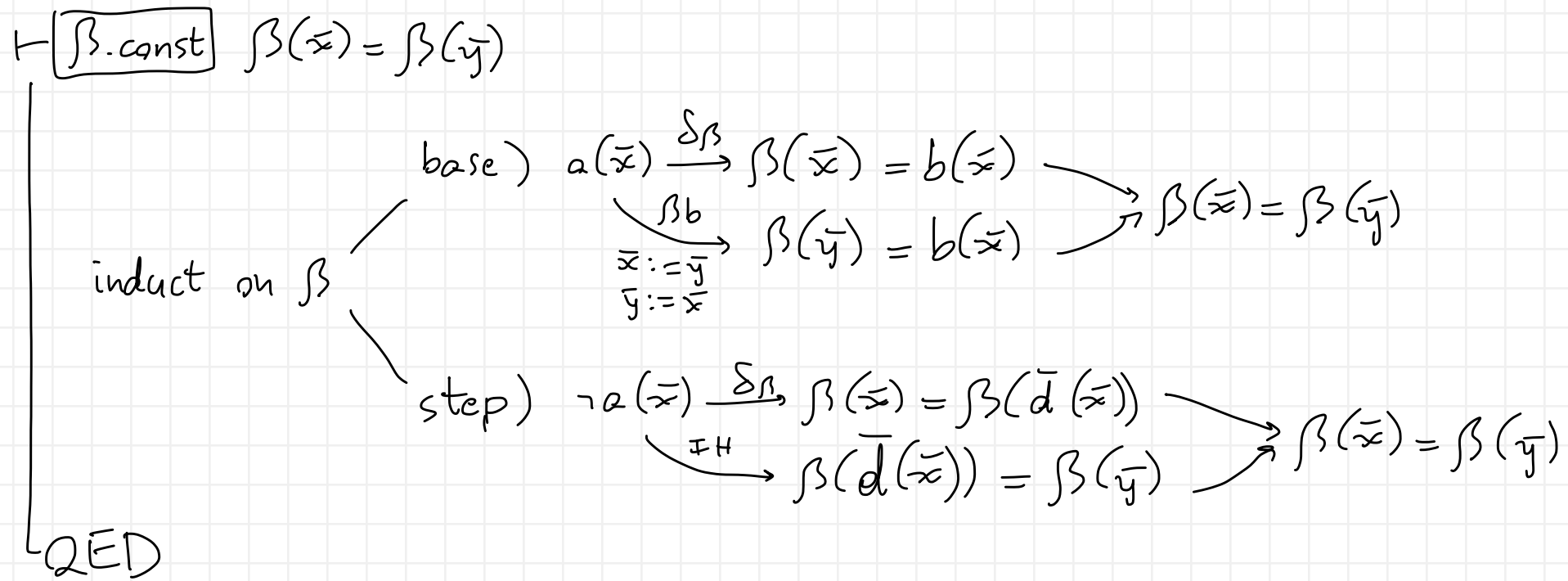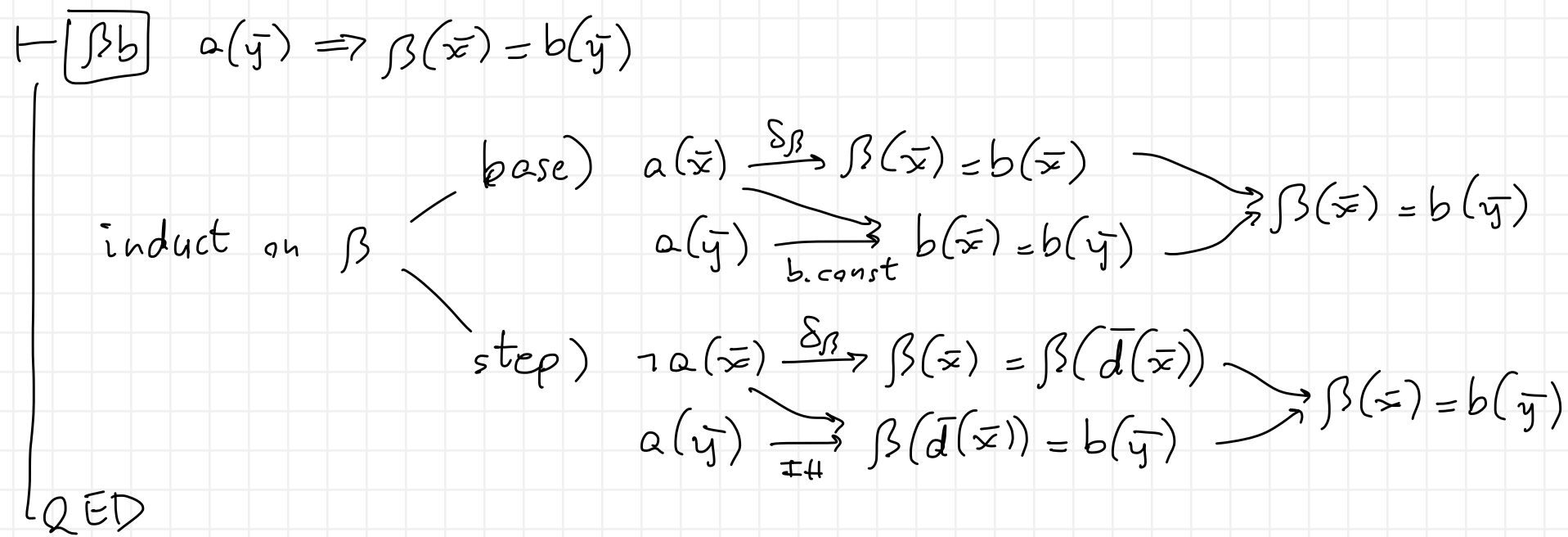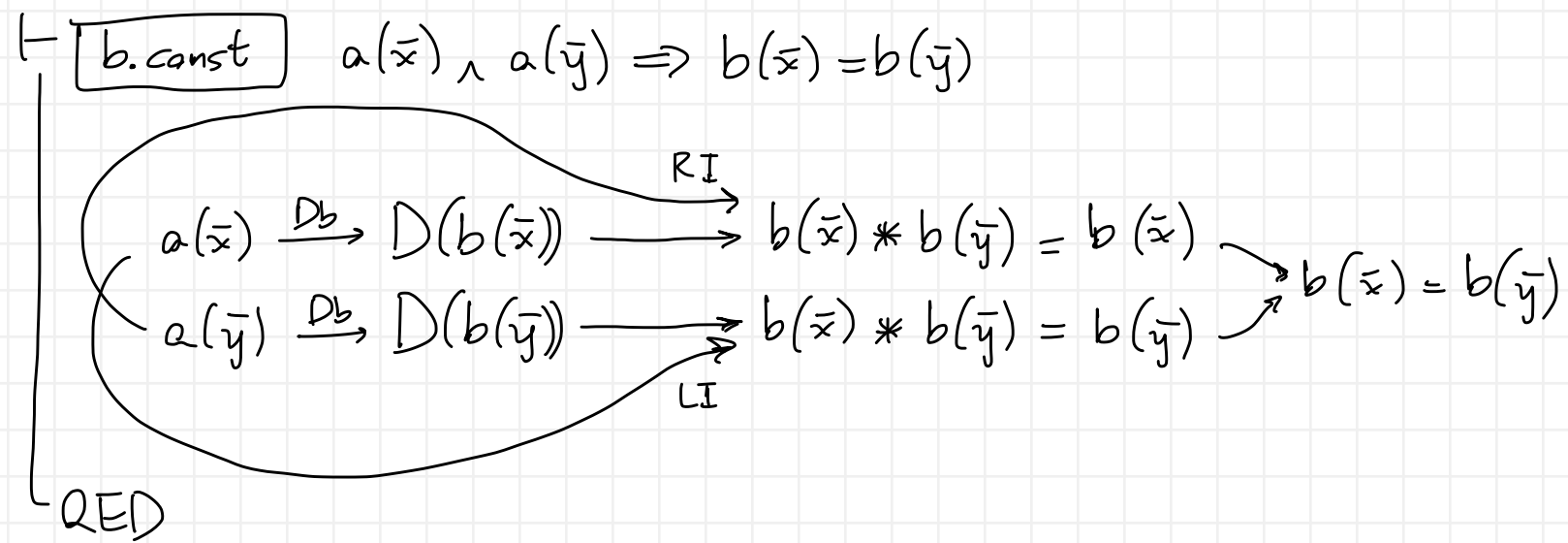
$D* \wedge Db_0 \wedge ASC \wedge LI_0 \wedge RI_0$

$\Downarrow$

$(D, *, b_0)$ is a monoid

Guards

old function: $f(\bar{x}) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\gamma_{\bar{d}}(\bar{x}) = \gamma_{d_1}(\bar{x}) \wedge \cdots \wedge \gamma_{d_n}(\bar{x})$

$\boxed{\sqrt{f}}\quad \gamma_{\gamma_f}(\bar{x}) \wedge \left[\gamma_f(\bar{x}) \Rightarrow \gamma_a(\bar{x}) \wedge \left[a(\bar{x}) \Rightarrow \gamma_b(\bar{x})\right] \wedge \left[\neg a(\bar{x}) \Rightarrow \gamma_c(\bar{x}) \wedge \gamma_{\bar{d}}(\bar{x}) \wedge \gamma_f(\bar{d}(\bar{x})) \wedge \gamma_*(c(\bar{x}), f(\bar{d}(\bar{x})))\right]\right]$

$D \subseteq \mathcal{U}\quad -\ \text{domain}$

conditions $\begin{cases} \boxed{GD}\quad \gamma_D = \mathcal{U} & -\ D\ \text{always well-defined} \\ \boxed{G*}\quad \gamma_* \supseteq D \times D & -\ *\ \text{well-defined at least over } D \end{cases}$

new function: $f'(\bar{x}, r) \triangleq \begin{cases} \underline{if}\ a(\bar{x})\ \underline{then}\ r & \underline{else}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) & \Leftarrow RI \\ \underline{if}\ a(\bar{x})\ \underline{then}\ r * b(\bar{x})\ \underline{else}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) & \Leftarrow \neg RI \end{cases}\quad -\ \text{as before}$

$\gamma_{f'}(\bar{x}, r) \triangleq \left[\gamma_f(\bar{x}) \wedge D(r)\right]$

$\vdash \boxed{\sqrt{f'}}\quad \omega_{f'}(\bar{x}, r)$

RI)

$\omega_{f'}(\bar{x}, r) = \underset{\sqrt{f}}{\gamma_{\gamma_f}(\bar{x})} \wedge \underset{GD}{\gamma_D(r)} \wedge \left[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \underset{\sqrt{f}}{\gamma_a(\bar{x})} \wedge \left[\neg a(\bar{x}) \Rightarrow \underset{\sqrt{f}}{\gamma_{\bar{d}}(\bar{x})} \wedge \underset{\sqrt{f}}{\gamma_c(\bar{x})} \wedge \underset{G*}{\gamma_*(r, c(\bar{x}))} \wedge \underset{\sqrt{f}}{\gamma_f(\bar{d}(\bar{x}))} \wedge \underset{D*}{D(r * c(\bar{x}))}\right]\right]$

$\underset{Dc}{D(c(\bar{x}))}$

¬RI)

$\omega_{f'}(\bar{x}, r) = \underset{\sqrt{f}}{\gamma_{\gamma_f}(\bar{x})} \wedge \underset{GD}{\gamma_D(r)} \wedge \left[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \underset{\sqrt{f}}{\gamma_a(\bar{x})} \wedge \left[a(\bar{x}) \Rightarrow \underset{\sqrt{f}}{\gamma_b(\bar{x})} \wedge \underset{G*}{\gamma_*(r, b(\bar{x}))}\right] \wedge \left[\neg a(\bar{x}) \Rightarrow \ldots\right]\right]$

$\underset{Db}{D(b(\bar{x}))}$

$\overset{\sqrt{f}\ Dc\ G*\ D*}{}$ — as with RI

QED

$\beta(\bar{x}) \triangleq \underline{if} \ a(\bar{x}) \ \underline{then} \ b(\bar{x}) \ \underline{else} \ \beta(\bar{d}(\bar{x}))$ — as before

$\gamma_\beta(\bar{x}) \triangleq \gamma_f(\bar{x})$

$\vdash \boxed{\sqrt{\beta}} \quad \omega_\beta(\bar{x})$

$\omega_\beta(\bar{x}) = \gamma_{\gamma_f}(\bar{x}) \wedge \left[\gamma_f(\bar{x}) \Rightarrow \gamma_a(\bar{x}) \wedge \left[a(\bar{x}) \Rightarrow \gamma_b(\bar{x}) \right] \wedge \left[\neg a(\bar{x}) \Rightarrow \gamma_{\bar{d}}(\bar{x}) \wedge \gamma_f(\bar{d}(\bar{x})) \right] \right]$

(with $\sqrt{f}$ annotations over the struck-through terms)

QED

wrapper: $\tilde{f}(\bar{x}) \triangleq \begin{cases} f'(\bar{x}, \beta(\bar{x})) & \Leftarrow \text{LI} \\ \underline{if} \ a(\bar{x}) \ \underline{then} \ b(\bar{x}) \ \underline{else} \ f'(\bar{d}(\bar{x}), c(\bar{x})) & \Leftarrow \neg \text{LI} \end{cases}$ — as before

$\gamma_{\tilde{f}}(\bar{x}) \triangleq \gamma_f(\bar{x})$

$\vdash \boxed{\sqrt{\tilde{f}}} \quad \omega_{\tilde{f}}(\bar{x})$

LI)

$\omega_{\tilde{f}}(\bar{x}) = \gamma_{\gamma_f}(\bar{x}) \wedge \left[\gamma_f(\bar{x}) \Rightarrow \gamma_\beta(\bar{x}) \wedge \gamma_{f'}(\bar{x}, \beta(\bar{x})) \right]$

(with $\sqrt{f}$, $\delta\gamma_\beta$, $D\beta$, $\delta\gamma_{f'}$ annotations)

$\neg$LI)

$\omega_{\tilde{f}}(\bar{x}) = \gamma_{\gamma_f}(\bar{x}) \wedge \left[\gamma_f(\bar{x}) \Rightarrow \gamma_a(\bar{x}) \wedge \left[a(\bar{x}) \Rightarrow \gamma_b(\bar{x}) \right] \wedge \left[\neg a(\bar{x}) \Rightarrow \gamma_{\bar{d}}(\bar{x}) \wedge \gamma_c(\bar{x}) \wedge \gamma_{f'}(\bar{d}(\bar{x}), c(\bar{x})) \right] \right]$

(with $\sqrt{f}$, $\delta\gamma_{f'}$, Dc, $D(c(\bar{x}))$ annotations)

QED

if $\tilde{f}$ is not generated, a proof like this establishes that the body of $\tilde{f}$ is guard-verified under $\gamma_f(\bar{x})$:
this theorem is useful to guard-verify terms where a call to f is replaced with the (instantiated) body of $\tilde{f}$

Decomposition of the Old Function

$f(\bar{x}) \triangleq$ if t_1 then t_2 else t_3 t_1, t_2, t_3 terms without let f occurs in t_3, not in t_1 or t_2

$a \triangleq \lambda \bar{x}. t_1$

$b \triangleq \lambda \bar{x}. t_2$

all calls to f in t_3 must be identical : $f(s_1, \dots, s_n)$ $s_1, \dots, s_n$ terms not containing f $n > 0$

$d_i \triangleq \lambda \bar{x}. s_i$

$\tilde{t}_3 \triangleq t_3[f(s_1, \dots, s_n)/r]$, r fresh variable $\begin{cases} r \in FV(\tilde{t}_3) \text{, otherwise } f \text{ would not be recursive} \\ \tilde{t}_3 \not\equiv r \text{, otherwise } f \text{ would be already tail-recursive} \end{cases}$

$C \triangleq \{(c, *) \in (\mathcal{U}^n \to \mathcal{U}) \times (\mathcal{U} \times \mathcal{U} \to \mathcal{U}) \mid \forall \bar{x}. \tilde{t}_3 = c(\bar{x}) * r\}$ —   candidates for c and $*$

    $C = \emptyset$, e.g. $\tilde{t}_3 \equiv g(x_1, x_2, r)$

    $|C| = 1$, e.g. $\tilde{t}_3 \equiv g(x_1, r)$, $C = \{(id, g)\}$

    $|C| > 1$, e.g. $\tilde{t}_3 \equiv g(h(x_1), r)$, $C = \{(h, g), (id, \lambda(q,r). g(h(q), r))\}$

exactly one $(c, *) \in C$ for each term s such that $r \notin FV(s)$ and $FV(\tilde{t}_3[s/q]) \subseteq \{q, r\}$, q fresh variable

    s includes all occurrences of $\bar{x}$ in $\tilde{t}_3$

    special case: $FV(\tilde{t}_3) = \{r\}$, i.e. $\bar{x}$ do not occur in $\tilde{t}_3$

       $\Rightarrow$ any s would do, but then $*$ may ignore its first argument, making ASC, LI, RI less likely

when two such terms s and s' are one a subterm of the other, one is not always better than the other, e.g.:

$\tilde{t}_3 \equiv -x_1 + r \begin{cases} s \equiv x_1 &\Rightarrow * = \lambda(q,r). -q + r \text{ is not associative} \\ s' \equiv -x_1 &\Rightarrow * = \lambda(q,r). q + r \text{ is associative} \end{cases} \Big) \Rightarrow$ larger term is better

$\tilde{t}_3 \equiv -(-x_1) + r \begin{cases} s \equiv x_1 &\Rightarrow * = \lambda(q,r). -(-q) + r \text{ is associative} \\ s' \equiv -x_1 &\Rightarrow * = \lambda(q,r). -q + r \text{ is not associative} \end{cases} \Big) \Rightarrow$ smaller term is better

Special Case : Ground Base Value

old function : $f(\bar{x}) \triangleq \underline{if}\ t_1\ \underline{then}\ t_2\ \underline{else}\ t_3$ — as in discussion about decomposition

$FV(t_2) = \emptyset$ — ground base value $\implies$ $b \triangleq \lambda \bar{x}. b_0$, $b_0 \in \mathcal{U}$ $\implies$ $f(\bar{x}) = \underline{if}\ a(\bar{x})\ \underline{then}\ b_0\ \underline{else}\ c(\bar{x}) \ast f(\bar{d}(\bar{x}))$

$\vdash \boxed{\beta_0}$ $\beta(\bar{x}) = b_0$

induct on β
- base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b_0$
- step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x})) = b_0$
  $\xrightarrow{IH} \beta(\bar{d}(\bar{x})) = b_0$

QED

$\vdash \boxed{Db_0}$ $D(b_0)$

$D\beta \longrightarrow D(\beta(\bar{x})) \xrightarrow{\beta_0} D(b_0)$

QED

$LI \implies \vdash \boxed{LI_0}$ $D(u) \implies b_0 \ast u = u$

$D(u) \xrightarrow{LI\beta} \beta(\bar{x}) \ast u = u \xrightarrow{\beta_0} b_0 \ast u = u$

QED

$RI \implies \vdash \boxed{RI_0}$ $D(u) \implies u \ast b_0 = u$

$D(u) \xrightarrow{RI\beta} u \ast \beta(\bar{x}) = u \xrightarrow{\beta_0} u \ast b_0 = u$

QED

$D\ast \wedge Db_0 \wedge ASC \wedge LI_0 \wedge RI_0 \implies (D, \ast, b_0)$ is a monoid

---

f' and $f'f$ are the same as before (they do not use β and LI)

$LI \implies \vdash \boxed{ff'_0}$ $f(\bar{x}) = f'(\bar{x}, b_0)$

$Db_0 \longrightarrow D(b_0)$
$f'f \xrightarrow{r := b_0} f'(\bar{x}, b_0) = b_0 \ast f(\bar{x}) = f(\bar{x})$
$Df \longrightarrow D(f(\bar{x})) \xrightarrow{LI_0}$

QED

wrapper : $\tilde{f}(\bar{x}) \triangleq f'(\bar{x}, b_0) \implies \vdash \boxed{f\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$ — as before

$$\alpha(\bar{x}) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ \bar{x}\ \underline{else}\ \alpha(\bar{d}(\bar{x})) \qquad \alpha: \mathcal{U}^n \to \mathcal{U}^n \qquad - \quad \text{calculate } \bar{x}_0 \text{ such that } a(\bar{x}_0) \text{ holds, from any } \bar{x}$$

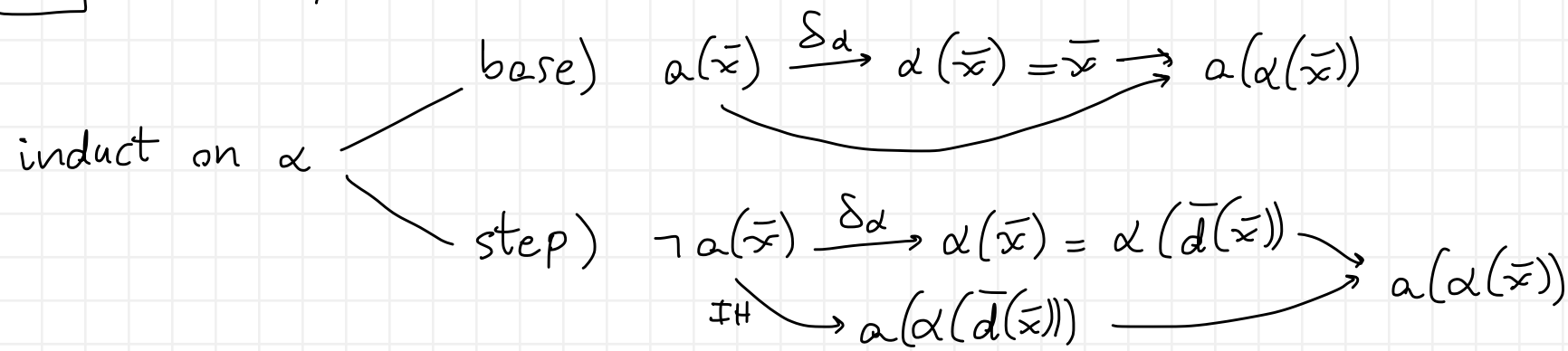$$\mu_\alpha(\bar{x}) \triangleq \mu_f(\bar{x}) \qquad \prec_\alpha \triangleq \prec_f$$

$\vdash \boxed{\tau_\alpha} \quad \neg\, a(\bar{x}) \implies \mu_\alpha(\bar{d}(\bar{x})) \prec_\alpha \mu_\alpha(\bar{x})$

$\qquad \neg\, a(\bar{x}) \xrightarrow{\tau_f} \mu_f(\bar{d}(\bar{x})) \prec_f \mu_f(\bar{x})$

$\qquad\qquad\qquad \delta_{\mu_\alpha} \| \qquad \delta_{\prec_\alpha} \| \qquad \| \delta_{\mu_\alpha}$

$\qquad\qquad\qquad \mu_\alpha(\bar{d}(\bar{x})) \prec_\alpha \mu_\alpha(\bar{x})$

$\llcorner$ QED

$\vdash \boxed{a\alpha} \quad a(\alpha(\bar{x}))$

$\text{induct on } \alpha$

$\qquad \text{base)}\quad a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \bar{x} \longrightarrow a(\alpha(\bar{x}))$

$\qquad \text{step)}\quad \neg\, a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \alpha(\bar{d}(\bar{x})) \searrow$
$\qquad\qquad\qquad \xrightarrow{IH} a(\alpha(\bar{d}(\bar{x}))) \longrightarrow a(\alpha(\bar{x}))$

$\llcorner$ QED

$\sqrt{f} \implies \vdash \boxed{\gamma_f \alpha} \quad \gamma_f(\bar{x}) \implies \gamma_f(\alpha(\bar{x}))$

$\text{induct on } \alpha$

$\qquad \text{base)}\quad a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \bar{x} \searrow$
$\qquad\qquad\qquad \gamma_f(\bar{x}) \longrightarrow \gamma_f(\alpha(\bar{x}))$

$\qquad \text{step)}\quad \neg\, a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \alpha(\bar{d}(\bar{x})) \searrow$
$\qquad\qquad \left(\gamma_f(\bar{x}) \xrightarrow{\sqrt{f}} \gamma_f(\bar{d}(\bar{x})) \xrightarrow{IH} \gamma_f(\alpha(\bar{d}(\bar{x}))) \right) \nearrow \gamma_f(\alpha(\bar{x}))$

$\llcorner$ QED
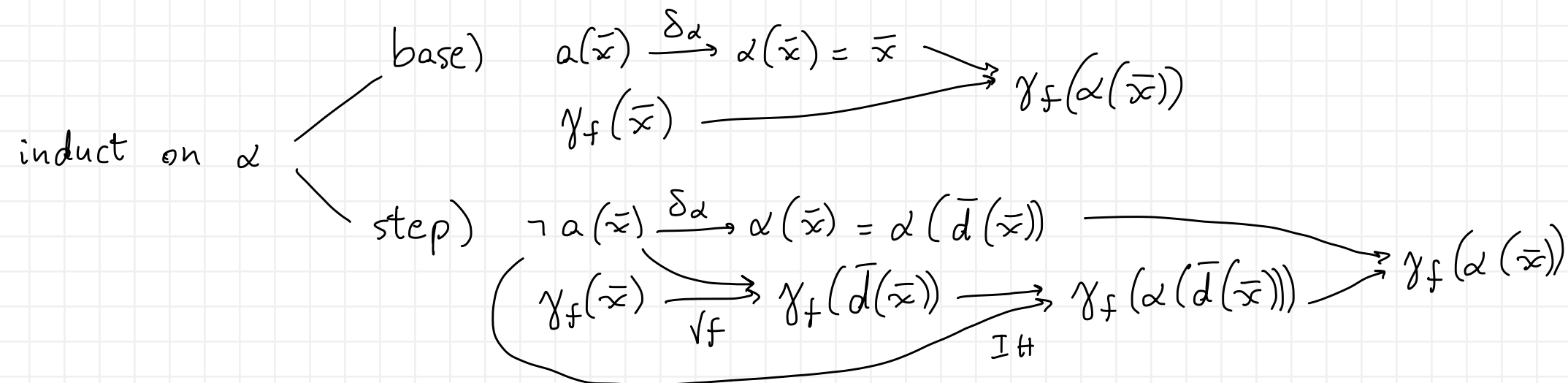
$b = \lambda \bar{x}. b_0 \implies \gamma_b \overset{\Delta}{=} \lambda \bar{x}. \gamma_{b_0}$, $\gamma_{b_0} \in \mathcal{U}$

$\vdash \boxed{\gamma_b. \text{const}}$ $\gamma_b(\bar{x}) = \gamma_b(\bar{y})$

$\gamma_b(\bar{x}) \underset{\delta\gamma_b}{=} \gamma_{b_0} \underset{\delta\gamma_b}{=} \gamma_b(\bar{y})$

$\llcorner$ QED

$\sqrt{f} \implies \vdash \boxed{Gb}$ $\gamma_f(\bar{x}) \implies \gamma_b(\bar{x})$

$\gamma_f(\bar{x}) \xrightarrow[\gamma_f \alpha]{} \gamma_f(\alpha(\bar{x})) \searrow$

$\qquad\qquad\qquad\qquad \overset{\sqrt{f}}{\searrow} \gamma_b(\alpha(\bar{x})) \overset{\gamma_b.\text{const}}{=\!=} \gamma_b(\bar{x})$

$a\alpha \longrightarrow a(\alpha(\bar{x})) \nearrow$

$\llcorner$ QED

$\sqrt{f} \wedge LI \implies \vdash \boxed{\sqrt{\tilde{f}}}$ $\omega_{\tilde{f}}(\bar{x})$

$\omega_{\tilde{f}}(\bar{x}) = \gamma_{\gamma_f}(\bar{x}) \wedge \left[\gamma_f(\bar{x}) \implies \gamma_b(\bar{x}) \wedge \gamma_{f'}(\bar{x}, b_0) \right] \qquad\qquad — \quad b_0 = b(\bar{x})$

with annotations: $\sqrt{f}$, Gb , $Db_0 \; \delta_{\gamma_{f'}}$

$\llcorner$ QED

summary of this special case (ground base value): b_0 can be used instead of β (when LI holds)

note that Db_0 and LI_0 can be proved without using β :

$\vdash \boxed{Db_0}$ $D(b_0)$ $\qquad$ — alternative proof

$a\alpha \longrightarrow a(\alpha(\bar{x})) \xrightarrow[\bar{x} := \alpha(\bar{x})]{Db} D(b(\alpha(\bar{x}))) \xrightarrow[\bar{x} := \alpha(\bar{x})]{\delta b} D(b_0)$

$\llcorner$ QED

$LI \implies \vdash \boxed{LI_0}$ $D(u) \implies b_0 * u = u$ $\qquad$ — alternative proof

$a\alpha \longrightarrow a(\alpha(\bar{x})) \xrightarrow[\bar{x} := \alpha(\bar{x})]{LI} b(\alpha(\bar{x})) * u = u \xrightarrow[\bar{x} := \alpha(\bar{x})]{\delta b} b_0 * u = u$

$D(u)$

$\llcorner$ QED

Extension of Operator Associativity and Closure outside the Domain

conditions $\begin{cases} \boxed{Db} & a(\bar{x}) \implies D(b(\bar{x})) \qquad \text{— as before} \\ \boxed{D*'} & D(u*v) \qquad \text{— unconditional version of } D* \\ \boxed{ASC'} & u*(v*w) = (u*v)*w \qquad \text{— unconditional version of ASC} \\ \boxed{LI} & a(\bar{x}) \wedge D(u) \implies b(\bar{x}) * u = u \qquad \text{— as before, but required (not optional)} \\ \boxed{RI} & a(\bar{x}) \wedge D(u) \implies u * b(\bar{x}) = u \qquad \text{— as before, and still optional} \end{cases}$

$\left[Db \wedge D*' \wedge ASC' \right]$ is neither weaker nor stronger than $\left[Db \wedge Dc \wedge D* \wedge ASC \right]$

$D*'$ and ASC' may be satisfied when $*$ fixes its arguments to be in D

$\vdash \boxed{Df} \quad D(f(\bar{x})) \qquad \text{— as before, but slightly different proof}$

$a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x}) \searrow$
$\qquad \xrightarrow{Db} D(b(\bar{x})) \longrightarrow D(f(\bar{x}))$

$\neg a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x})) \searrow D(f(\bar{x}))$
$\qquad D*' \xrightarrow[\substack{u:=c(\bar{x}) \\ v:=f(\bar{d}(\bar{x}))}]{} D(c(\bar{x}) * f(\bar{d}(\bar{x}))) \nearrow D(f(\bar{x}))$

QED
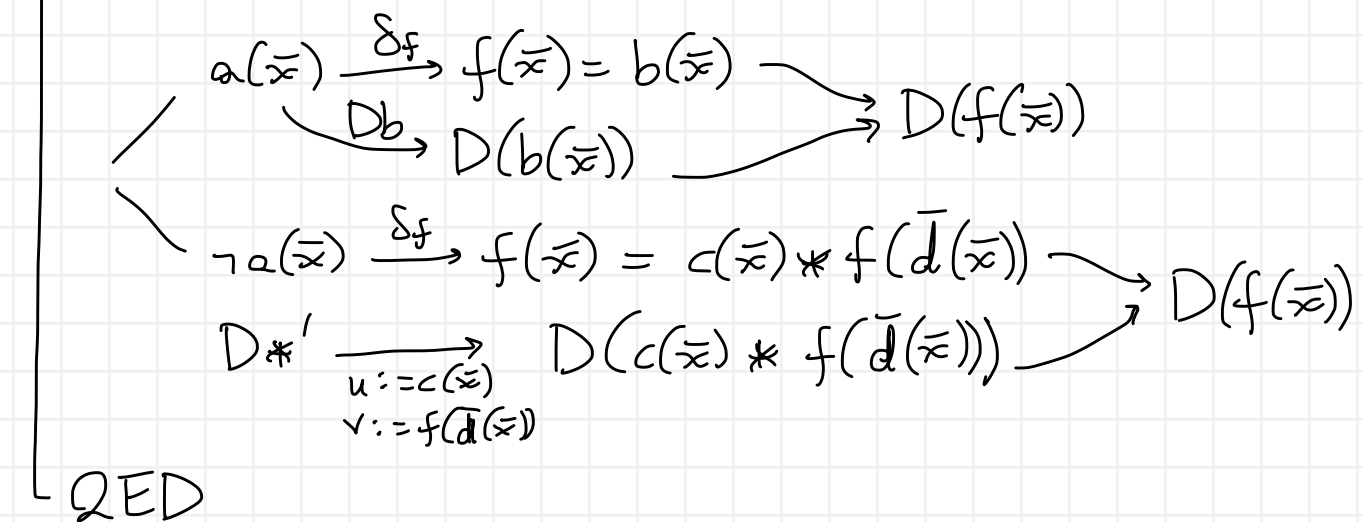
$\beta(\bar{x}) \triangleq \underline{if}\ a(\bar{x})\ \underline{then}\ b(\bar{x})\ \underline{else}\ \beta(\bar{d}(\bar{x})) \qquad \text{— as before}$

$\vdash \boxed{D\beta} \quad D(\beta(\bar{x})) \qquad \text{— as before, same proof}$

$\vdash \boxed{LI\beta} \quad D(u) \implies \beta(\bar{x}) * u = u \qquad \text{— as before, same proof}$

$RI \implies \vdash \boxed{RI\beta} \quad D(u) \implies u * \beta(\bar{x}) = u \qquad \text{— as before, same proof}$

$RI \implies (D, *, b_0) \text{ is a monoid} \qquad \text{— as before, same lemmas and proofs}$

new function: $f'(\bar{x},r) \triangleq \begin{cases} \text{if } a(\bar{x}) \text{ then } r \quad\text{ else } f'(\bar{d}(\bar{x}),r*c(\bar{x})) & \Leftarrow RI \\ \text{if } a(\bar{x}) \text{ then } r*b(\bar{x}) \text{ else } f'(\bar{d}(\bar{x}),r*c(\bar{x})) & \Leftarrow \neg RI \end{cases}$ — as before

$\vdash$ $\boxed{f'f}$ $D(r) \Rightarrow f'(\bar{x},r) = r * f(\bar{x})$ — as before, but slightly different proof

induct on f'

base)

   RI) $a(\bar{x}) \overset{\delta_f}{\nearrow} f(\bar{x}) = b(\bar{x})$
         $\overset{\delta_{f'}}{\searrow} f'(\bar{x},r) = r \; = r*b(x) = r * f(\bar{x})$

       $D(r) \underset{RI}{\longrightarrow} r * b(\bar{x}) = r$

   $\neg RI)$ $a(\bar{x}) \overset{\delta_f}{\nearrow} f(\bar{x}) = b(\bar{x})$
         $\overset{\delta_{f'}}{\searrow} f'(\bar{x},r) = r*b(x) = r * f(\bar{x})$

step) $\neg a(\bar{x}) \overset{\delta_f}{\nearrow} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
         $\overset{\delta_{f'}}{\searrow} f'(\bar{x},r) = f'(\bar{d}(\bar{x}), r*c(\bar{x})) = (r*c(\bar{x}))*f(\bar{d}(\bar{x})) \overset{ASC'}{=} r*(c(\bar{x})*f(\bar{d}(\bar{x}))) = r * f(\bar{x})$

     IH $\Bigg($ $D*' \longrightarrow D(r*c(\bar{x}))$

        $D(r*c(\bar{x})) \Rightarrow f'(\bar{d}(\bar{x}), r*c(\bar{x})) = (r*c(\bar{x}))*f(\bar{d}(\bar{x}))$
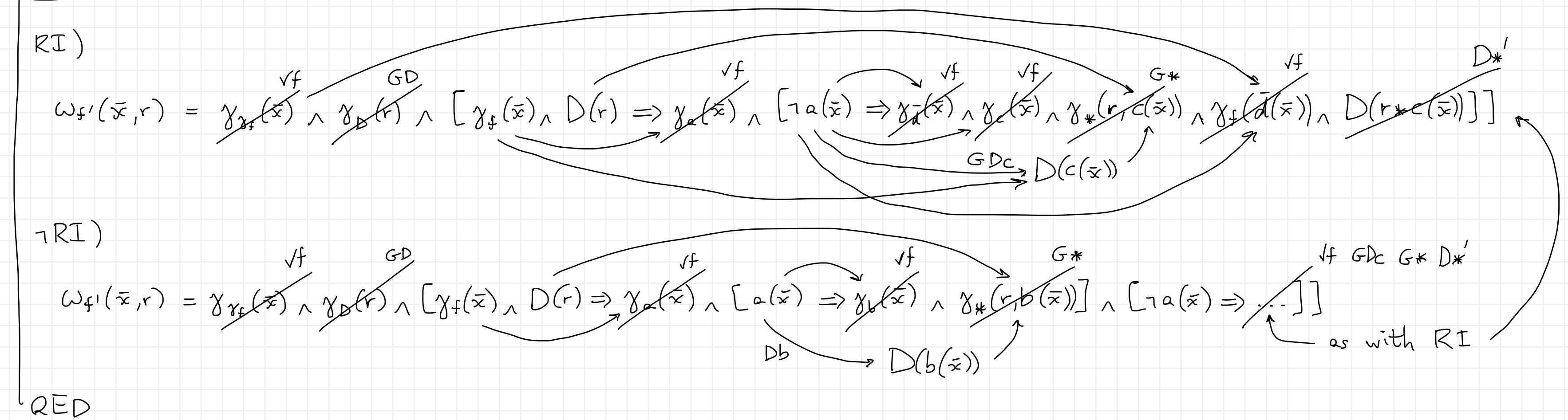
QED
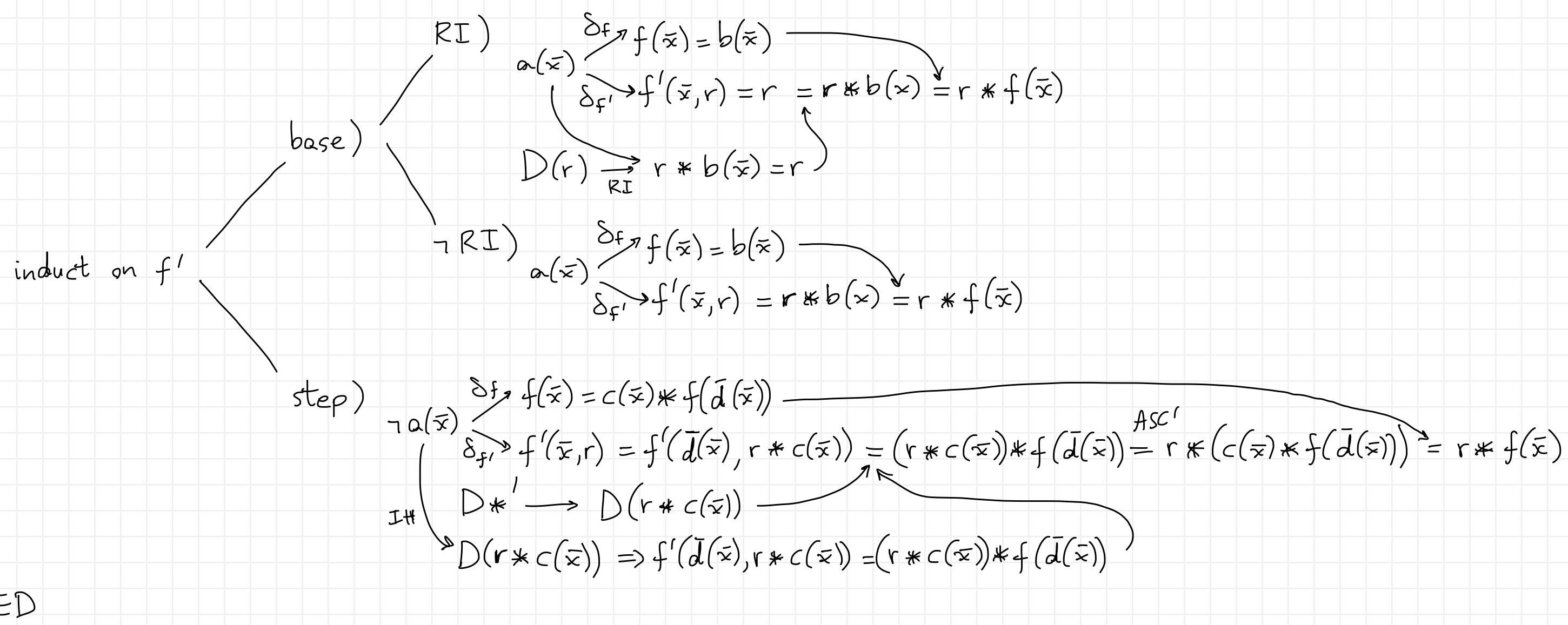
$\vdash$ $\boxed{ff'}$ $f(\bar{x}) = f'(\bar{x}, \beta(\bar{x}))$ — as before, same proof

the proof of ff' when LI does not hold needs Dc to use an instance of $f'f$
$\Rightarrow$ LI is required here; otherwise $(Db \wedge Dc \wedge D*' \wedge ASC')$ is stronger (i.e. less satisfiable) than $(Db \wedge Dc \wedge D* \wedge ASC)$

conditions $\begin{cases} \boxed{GD} & \gamma_D = \mathcal{U} & \text{— as before} \\ \boxed{G*} & \gamma_* \supseteq D \times D & \text{— as before} \\ \boxed{GDc} & \gamma_f(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow D(c(\bar{x})) & \text{— weaker version of } Dc \text{, conditioned by the guard of } f \end{cases}$

$\gamma_{f'}(\bar{x}, r) \triangleq \left[\gamma_f(\bar{x}) \wedge D(r) \right]$ — as before

$\vdash \boxed{\checkmark f'} \quad \omega_{f'}(\bar{x}, r)$ — as before, but slightly different proof

RI)

$\omega_{f'}(\bar{x}, r) = \overset{\checkmark f}{\cancel{\gamma_{f'}(\bar{x})}} \wedge \overset{GD}{\cancel{\gamma_D(r)}} \wedge \left[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \overset{\checkmark f}{\cancel{\gamma_a(\bar{x})}} \wedge \left[\neg a(\bar{x}) \Rightarrow \overset{\checkmark f}{\cancel{\gamma_{\bar{a}}(\bar{x})}} \wedge \overset{\checkmark f}{\cancel{\gamma_c(\bar{x})}} \wedge \overset{G*}{\cancel{\gamma_*(r, c(\bar{x}))}} \wedge \overset{\checkmark f}{\cancel{\gamma_f(\bar{d}(\bar{x}))}} \wedge \overset{D*'}{D(r * c(\bar{x}))} \right] \right]$

$\overset{GDc}{\longrightarrow} D(c(\bar{x}))$

¬RI)

$\omega_{f'}(\bar{x}, r) = \overset{\checkmark f}{\cancel{\gamma_{f'}(\bar{x})}} \wedge \overset{GD}{\cancel{\gamma_D(r)}} \wedge \left[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \overset{\checkmark f}{\cancel{\gamma_a(\bar{x})}} \wedge \left[a(\bar{x}) \Rightarrow \overset{\checkmark f}{\cancel{\gamma_b(\bar{x})}} \wedge \overset{G*}{\cancel{\gamma_*(r, b(\bar{x}))}} \right] \wedge \left[\neg a(\bar{x}) \Rightarrow \overset{\checkmark f\ GDc\ G*\ D*'}{\cancel{\ldots}} \right] \right]$

$\overset{Db}{\longrightarrow} D(b(\bar{x}))$

— as with RI

QED

$\gamma_\beta(\bar{x}) \triangleq \gamma_f(\bar{x})$ — as before

$\vdash \boxed{\checkmark \beta}$ — as before, same proof

$\gamma_{\tilde{f}}(\bar{x}) \triangleq \gamma_f(\bar{x})$ — as before

$\vdash \boxed{\checkmark \tilde{f}}$ — as before, same proof (only LI case)

special case of a ground base value : everything is the same as before

Associative-Only Variant, Extended outside the Domain

conditions $\begin{cases} \boxed{D*} \quad D(u) \wedge D(v) \implies D(u*v) & \text{— conditional is weaker, more applicable} \\ \boxed{ASC'} \quad u*(v*w) = (u*v)*w & \text{— unconditional version of ASC} \end{cases}$

$\vdash \boxed{f'f} \quad f'(\bar{x}, r) = r * f(\bar{x})$ — no $D(r)$ condition

induct f'

base) $a(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = r * b(\bar{x}) = r * f(\bar{x})$
$\phantom{base) a(\bar{x})} \xrightarrow{\delta_f} f(\bar{x}) = b(x)$

step) $\neg a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
$\phantom{step) \neg a(\bar{x})} \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) \overset{IH}{=} (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \overset{ASC'}{=} r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x})$

QED

$\vdash \boxed{ff'} \quad f(\bar{x}) = \text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } f'(\bar{d}(\bar{x}), c(\bar{x}))$

$f(\bar{x}) \overset{\delta_f}{=} \text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } c(\bar{x}) * f(\bar{d}(\bar{x})) = \text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } f'(\bar{d}(\bar{x}), c(\bar{x}))$

$f'f \xrightarrow[\bar{x} := \bar{d}(\bar{x})]{r := c(\bar{x})} f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
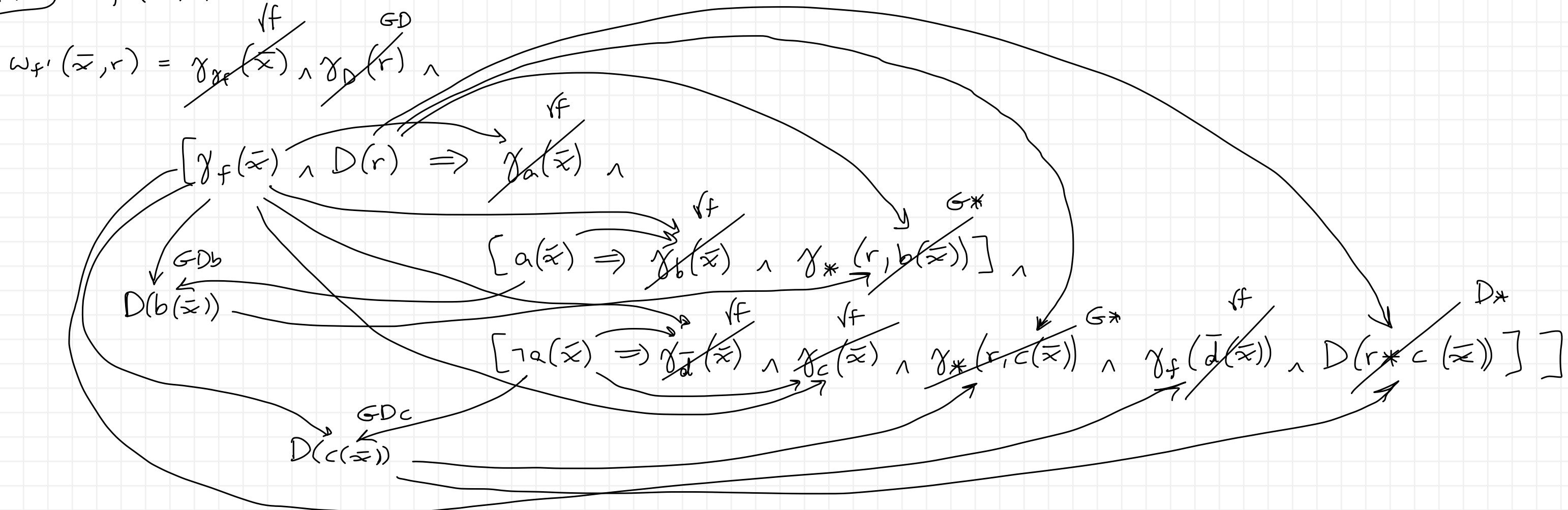
QED

$\vdash \boxed{f\tilde{f}} \quad f(\bar{x}) = \tilde{f}(\bar{x})$ — as before

conditions

$$\boxed{GD} \quad \gamma_D = \mathcal{U} \qquad \text{— as before}$$

$$\boxed{G*} \quad \gamma_* \supseteq D \times D \qquad \text{— as before}$$

$$\boxed{GDb} \quad \gamma_f(\bar{x}) \wedge a(\bar{x}) \Rightarrow D(b(\bar{x})) \qquad \text{— weaker version of Db, conditioned by the guard of } f$$

$$\boxed{GDc} \quad \gamma_f(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow D(c(\bar{x})) \qquad \text{— weaker version of Dc, conditioned by the guard of } f$$

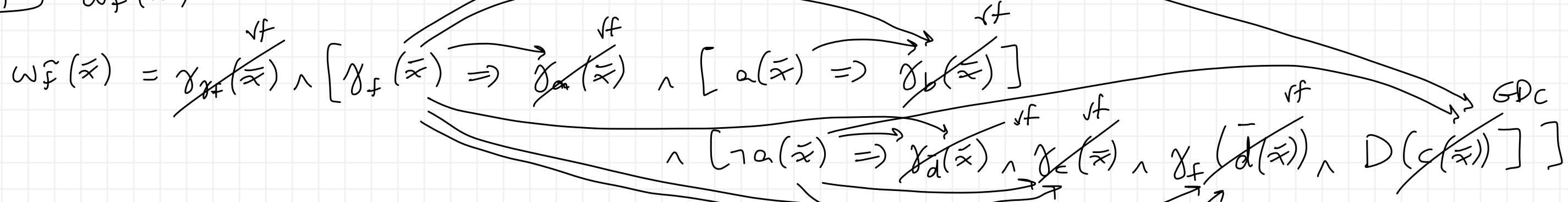$$\gamma_{f'}(\bar{x}, r) \triangleq [\gamma_f(\bar{x}) \wedge D(r)] \qquad \text{— as before}$$

$$\gamma_{\tilde{f}}(\bar{x}) \triangleq \gamma_f(\bar{x}) \qquad \text{— as before}$$

$\vdash \boxed{\langle f' \rangle} \quad \omega_{f'}(\bar{x}, r)$

$$\omega_{f'}(\bar{x}, r) = \gamma_{\tilde{f}}(\bar{x}) \wedge \gamma_D(r) \wedge$$

$$[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \gamma_a(\bar{x}) \wedge$$

$$[a(\bar{x}) \Rightarrow \gamma_b(\bar{x}) \wedge \gamma_*(r, b(\bar{x}))] \wedge$$

$$[\neg a(\bar{x}) \Rightarrow \gamma_d(\bar{x}) \wedge \gamma_c(\bar{x}) \wedge \gamma_*(r, c(\bar{x})) \wedge \gamma_f(\bar{d}(\bar{x})) \wedge D(r * c(\bar{x}))]]$$

(with: $D(b(\bar{x}))$ via GDb, $D(c(\bar{x}))$ via GDc)

QED

$\vdash \boxed{\langle \tilde{f} \rangle} \quad \omega_{\tilde{f}}(\bar{x})$

$$\omega_{\tilde{f}}(\bar{x}) = \gamma_{\tilde{f}}(\bar{x}) \wedge [\gamma_f(\bar{x}) \Rightarrow \gamma_a(\bar{x}) \wedge [a(\bar{x}) \Rightarrow \gamma_b(\bar{x})]$$

$$\wedge [\neg a(\bar{x}) \Rightarrow \gamma_d(\bar{x}) \wedge \gamma_c(\bar{x}) \wedge \gamma_f(\bar{d}(\bar{x})) \wedge D(c(\bar{x}))]]$$

QED